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Effective Results and Methods for Diophantine Equations over Finitely Generated Domains

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To our families

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Preface

This book is devoted to Diophantine equations where the solutions are taken from an integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$, which is a domain of the shape $\mathbb{Z}[z_1, \ldots, z_r]$ with a quotient field of characteristic 0, where the generators $z_1, \ldots, z_r$ may be algebraic or transcendental over $\mathbb{Q}$. For instance, the ring of integers and the rings of S-integers of a number field are finitely generated domains where all generators are algebraic. Our aim is to prove effective finiteness results for certain classes of Diophantine equations, i.e., results that not only show that the equations from the said classes have only finitely many solutions, but whose proofs provide methods to determine the solutions in principle.

There is an extensive literature on Diophantine equations with solutions taken from the ring of rational integers $\mathbb{Z}$, or from more general domains, containing theorems on the finiteness of the set of solutions of such equations. Most of the finiteness theorems over $\mathbb{Z}$, and more generally over rings of integers and S-integers of number fields are ineffective. Their proofs are mainly based on techniques from Diophantine approximation (e.g., the Thue–Siegel–Roth–Schmidt theory) often combined with algebra and arithmetic geometry. These techniques yield the finiteness of the number of solutions but do not enable one to determine the solutions. Lang (1960) and others used certain specialization arguments to extend several ineffective finiteness results to the even more general case when the solutions are taken from an arbitrary integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$.

Since the 1960s, a great number of ineffective finiteness theorems over number fields were made effective and new theorems were obtained in effective form by means of A. Baker's effective theory of logarithmic forms. These results give effective upper bounds for the solutions, and thereby make it possible, at least in principle, to find all the solutions of the equations under consideration. Analogous theorems were established by Mason (1984) and others over function fields of characteristic 0 as well, which provide effective upper bounds for the heights of the solutions, but do not imply the finiteness of the number of solutions.

Győry (1983, 1984b) was the first to extend effective Diophantine results over number fields to the finitely generated case and proved effective finiteness theorems over certain restricted classes of finitely generated integral domains over $\mathbb{Z}$ of zero characteristic. He developed an effective specialization method, reducing the initial equations to the number field and function field cases, and using the corresponding effective results over number fields and function fields, he derived effective bounds for the solutions of the initial equations.

In the paper Evertse and Győry (2013), Győry's specialization method was extended to the case of arbitrary finitely generated domains of characteristic 0 over $\mathbb{Z}$. The crucial new tool in this extension was the work of Aschenbrenner (2004) on effective commutative algebra. Evertse's and Győry's general specialization method may be viewed as a "machine," which takes as input an effective Diophantine finiteness result concerning S-integral solutions over number fields together with an effective analogue over function fields, and produces as output a corresponding effective result over finitely generated domains. This general specialization method led to effective finiteness results for various classes of Diophantine equations over arbitrary domains of characteristic 0 that are finitely generated over $\mathbb{Z}$: Evertse and Győry (2013, 2014, 2015), Bérczes, Evertse, and Győry (2014), Bérczes (2015a, 2015b), and Koymans (2016, 2017) established general effective finiteness theorems over finitely generated domains of characteristic 0 for several classical equations, including *unit equations in two unknowns, Thue equations, hyper- and superelliptic equations*, and the *Catalan equation*. An important feature of these results is their quantitative nature, i.e., they give upper bounds for the sizes (suitable measures) of the solutions in terms of defining parameters for the domain from which the solutions are taken and for the Diophantine equation under consideration.

Our book provides the first comprehensive treatment of effective results and methods for Diophantine equations over finitely generated domains. Similarly to the above-mentioned literature, most of the results in our book are proved in quantitative form, giving effective bounds for the sizes of the solutions. Apart from the results mentioned above, our book contains new material, concerning *decomposable form equations* over finitely generated domains. Here, we have adapted the method of Győry (1973, 1980a) and Győry and Papp (1978) to reduce the decomposable form equations under consideration to systems of unit equations in two unknowns. Here again, we give effective upper bounds for the sizes of the solutions, and for this purpose, we had to work out new effective procedures. As a special case, we get back the results on *discriminant equations* from Evertse and Győry (2017a, 2017b).

We believe that the results in this book do not exhaust the possibilities of our techniques. Hopefully, they will inspire further investigations to obtain new effective results for other classes of Diophantine equations over finitely generated domains.

This book is aimed at anyone (graduate student and expert) with basic knowledge of algebra (groups, commutative rings, fields, Galois theory) and elementary algebraic number theory. No further specialized knowledge of commutative algebra or algebraic geometry is presupposed.

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Glossary of Frequently Used Notation

General Notation

$\lvert\mathcal{A}\rvert$	cardinality of a finite set $\mathcal{A}$
$\log^* x$	$\max(1, \log x)$, $\log^* 0 := 1$
$\ll$, $\gg$	Vinogradov symbols; $A(x) \ll B(x)$ or $B(x) \gg A(x)$ means that there is a constant $c > 0$ such that $\lvert A(x)\rvert \leq cB(x)$ for all x in the specified domain. The constant c may depend on certain specified parameters independent of x
$\ll_{a,b,\dots}$	the positive constants implied by $\ll_{a,b,\dots}$ depend only on $a, b, \dots$ and are effectively computable
$O(\cdot)$	$c\times$ the expression between the parentheses, where c is an effectively computable positive absolute constant. The c may be different at each occurrence of $O(\cdot)$
$\mathbb{Z}$, $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$	integers, positive integers, non-negative integers
$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	rational numbers, real numbers, complex numbers
gcd	greatest common divisor
$D(f)$	discriminant of a polynomial $f(X)$
$\overline{K}$	algebraic closure of a field K
A	integral domain (i.e., commutative ring with 1 and without divisors of 0)
A^*	unit group (multiplicative group of invertible elements) of A
A_G	integral closure of A in an extension G of the quotient field of A
$A[X_1, \dots, X_n]$	ring of polynomials in n variables with coefficients in A

$A[\alpha_1,\dots,\alpha_n]$	$\{f(\alpha_1,\dots,\alpha_n) : f \in A[X_1,\dots,X_r]\}$, A-algebra generated by $\alpha_1,\dots,\alpha_n$
$\xi + \mathcal{M}$	$\{\xi + \eta : \eta \in \mathcal{M}\}$, $\mathcal{M}$-coset, where $\mathcal{M}$ is an A-module and ξ belongs to an A-module containing $\mathcal{M}$
$\mathcal{M}'/\mathcal{M}$	quotient A-module of two A-modules $\mathcal{M}',\mathcal{M}$, where $\mathcal{M}' \supseteq \mathcal{M}$; $\mathcal{M}'/\mathcal{M}$ consists of the $\mathcal{M}$-cosets $\xi + \mathcal{M}$ with $\xi \in \mathcal{M}'$, and is endowed with addition $(\xi_1 + \mathcal{M}) + (\xi_2 + \mathcal{M}) := (\xi_1 + \xi_2) + \mathcal{M}$ and scalar multiplication $a \cdot (\xi + \mathcal{M}) := a\xi + \mathcal{M}$, for $\xi_1,\xi_2,\xi \in \mathcal{M}'$ and $a \in A$
$H(Q)$, $L(Q)$	maximum of the absolute values resp. the sum of the absolute values of the coefficients of $Q \in \mathbb{Z}[X_1,\dots,X_n]$
$\deg Q$, $h(Q)$	the total degree of $Q \in \mathbb{Z}[X_1,\dots,X_n]$, resp. the logarithmic height $\log H(Q)$ of Q
$s(Q)$	$\max(1, \deg Q, h(Q))$, the size of Q

Finite Étale Algebras over Fields

Ω/K	finite étale algebra over a field K, i.e., a direct product $L_1 \times \cdots \times L_q$ of finite separable field extensions of K
$[\Omega : K]$	$\dim_K \Omega$
$x \mapsto x^{(i)}$	nontrivial K-algebra homomorphisms $\Omega \to \overline{K}$
$D_{\Omega/K}(\alpha)$	discriminant of $\alpha \in \Omega$ over K
A_Ω	integral closure of an integral domain A with quotient field K in a finite étale K-algebra Ω
$\mathcal{O}$	A-order of Ω, i.e., a subring of A_Ω containing A and generating Ω as a K-vector space

Algebraic Number Fields

$\mathrm{ord}_p(a)$	exponent of a prime number p in the unique prime factorization of $a \in \mathbb{Q}$, and $\mathrm{ord}_p(0) = \infty$
$\lvert a\rvert_p$	$p^{-\mathrm{ord}_p(a)}$, p-adic absolute value of $a \in \mathbb{Q}$
$\lvert a\rvert_\infty$	$\max(a, -a)$, ordinary absolute value of $a \in \mathbb{Q}$
$\mathbb{Q}_p$	p-adic completion of $\mathbb{Q}$, $\mathbb{Q}_\infty = \mathbb{R}$
$M_{\mathbb{Q}}$	$\{\infty\} \cup \{\text{primes}\}$, set of places of $\mathbb{Q}$
$\mathcal{O}_K$, D_K, h_K, R_K	ring of integers, discriminant, class number, regulator of a number field K
$\mathfrak{p}$, $\mathfrak{a}$	nonzero prime ideal, fractional ideal of $\mathcal{O}_K$

$[\alpha] = \alpha\mathcal{O}_K$	fractional ideal generated by α
$\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})$	exponent of $\mathfrak{p}$ in the unique prime ideal factorization of $\mathfrak{a}$
$\mathrm{ord}_{\mathfrak{p}}(\alpha)$	exponent of $\mathfrak{p}$ in the unique prime ideal factorization of (α) for $\alpha \in K$, with $\mathrm{ord}_{\mathfrak{p}}(0) := \infty$.
$N_K(\mathfrak{a})$	absolute norm of a fractional ideal $\mathfrak{a}$ of $\mathcal{O}_K$ (written as $N(\mathfrak{a})$ if it is clear which is the underlying number field)
$\mathcal{M}_K$	set of places of a number field K
$\lvert\cdot\rvert_v$ $(v \in \mathcal{M}_K)$	normalized absolute values of K, satisfying the product formula, with $\lvert\alpha\rvert_v := N_K(\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(\alpha)}$ if $\alpha \in K$ and $\mathfrak{p}$ is the prime ideal of $\mathcal{O}_K$ corresponding to the finite place v
K_v	completion of K at v
S_∞	set of infinite (archimedean) places
S	finite set of places of K, containing S_∞
$\mathcal{O}_S$	$\{\alpha \in K : \lvert\alpha\rvert_v \le 1 \text{ for } v \in \mathcal{M}_K\setminus S\}$, ring of S-integers, written as $\mathbb{Z}_S$ if $K = \mathbb{Q}$
$\mathcal{O}_S^*$	$\{\alpha \in K : \lvert\alpha\rvert_v = 1 \text{ for } v \in \mathcal{M}_K\setminus S\}$, group of S-units, written as $\mathbb{Z}_S^*$ if $K = \mathbb{Q}$
$N_S(\alpha)$	$\prod_{v\in S} \lvert\alpha\rvert_v$, S-norm of $\alpha \in K$
R_S	S-regulator
P_S, Q_S	$\max\{N_K(\mathfrak{p}_1),\dots,N_K(\mathfrak{p}_t)\}$, $\prod_{i=1}^t N_K(\mathfrak{p}_i)$, where $\mathfrak{p}_1,\dots,\mathfrak{p}_t$ are the prime ideals of $\mathcal{O}_K$ corresponding to the finite places of S
$\lvert\mathbf{x}\rvert_v$ $(v \in \mathcal{M}_K)$	$\max_i \lvert x_i\rvert_v$, v-adic norm of $\mathbf{x} = (x_1,\dots,x_n) \in K^n$
$H^{\mathrm{hom}}(\mathbf{x})$	$\left(\prod_{v\in\mathcal{M}_K} \lvert\mathbf{x}\rvert_v\right)^{1/[K:\mathbb{Q}]}$, absolute homogeneous height of $\mathbf{x} \in K^n$
$H(\mathbf{x})$	$\left(\prod_{v\in\mathcal{M}_K} \max(1,\lvert\mathbf{x}\rvert_v)\right)^{1/[K:\mathbb{Q}]}$, absolute height of $\mathbf{x} \in K^n$
$H(\alpha)$	$\left(\prod_{v\in\mathcal{M}_K} \max(1,\lvert\alpha\rvert_v)\right)^{1/[K:\mathbb{Q}]}$, absolute height of $\alpha \in K$
$h^{\mathrm{hom}}(\mathbf{x})$, $h(\mathbf{x})$, $h(\alpha)$	$\log H^{\mathrm{hom}}(\mathbf{x})$, $\log H(\mathbf{x})$, $\log H(\alpha)$, absolute logarithmic heights $(\mathbf{x} \in K^n, \alpha \in K)$
$h(P)$	$h(\mathbf{x}_P)$, $\mathbf{x}_P$ vector consisting of the nonzero coefficients of a polynomial $P \in K[X_1,\dots,X_n]$

Function Fields

$\Bbbk$	field of constants (always algebraically closed)
$\Bbbk((z))$	field of Laurent series in z

$g_{K/\Bbbk}$	genus of function field K with constant field $\Bbbk$ ($K/\Bbbk$ is always assumed to be of transcendence degree 1)
$\mathcal{M}_K$	set of (normalized discrete) valuations of K, trivial on $\Bbbk$
$v(\mathbf{x})$ $(v \in \mathcal{M}_K)$	$\min_i v(x_i)$, v-adic norm of $\mathbf{x} = (x_1, \dots, x_n) \in K^n$
$H_K^{\mathrm{hom}}(\mathbf{x})$	$-\sum_{v \in \mathcal{M}_K} v(\mathbf{x})$, homogeneous height of $\mathbf{x} \in K^n$
$H_K(x)$	$\sum_{v \in \mathcal{M}_K} \max(0, -v(x))$, height of $x \in K$
S	a finite subset of $\mathcal{M}_K$
$\mathcal{O}_S$	$\{\alpha \in K : v(\alpha) \geq 0 \text{ for } v \in \mathcal{M}_K \setminus S\}$, ring of S-integers
$\mathcal{O}_S^*$	$\{\alpha \in K : v(\alpha) = 0 \text{ for } v \in \mathcal{M}_K \setminus S\}$, group of S-units

Finitely Generated Domains

$A = \mathbb{Z}[z_1, \dots, z_r]$	$\{f(z_1, \dots, z_r) : f \in \mathbb{Z}[X_1, \dots, X_r]\}$, finitely generated integral domain over $\mathbb{Z}$ with quotient field $K = \mathbb{Q}(z_1, \dots, z_r)$
$A \simeq \mathbb{Z}[X_1, \dots, X_r]/\mathcal{I}$	$\mathcal{I} := \{f \in \mathbb{Z}[X_1, \dots, X_r] : f(z_1, \dots, z_r) = 0\}$, finitely generated ideal in $\mathbb{Z}[X_1, \dots, X_r]$
$\mathcal{I} = (f_1, \dots, f_M)$	ideal representation for A
$\tilde{\alpha} \in \mathbb{Z}[X_1, \dots, X_r]$	representative for $\alpha \in A$ if $\alpha = \tilde{\alpha}(z_1, \dots, z_r)$
A effectively given	if an ideal representation $(f_1, \dots, f_M)$ for A is given
$\alpha \in A$ effectively given (computable)	if a representative for α is given (can be computed)
$\{z_1 = X_1, \dots, z_q = X_q\}$	transcendence basis for $K = \mathbb{Q}(z_1, \dots, z_r)$ over $\mathbb{Q}$
$A_0 = \mathbb{Z}[X_1, \dots, X_q]$	subring of A with unique factorization
$\deg \alpha$, $h(\alpha)$ for $\alpha \in A_0$	the total degree and logarithmic height of α
$K_0 = \mathbb{Q}(X_1, \dots, X_q)$	quotient field of A_0
$K = K_0(w)$	where $w \in A$, integral over A_0 with degree D over K_0
$\overline{\deg}\, \alpha$ $(\alpha \in K)$	$\max(\deg P_{\alpha,0}, \dots, \deg P_{\alpha,D-1}, \deg Q_\alpha)$, where $P_{\alpha,0}, \dots, P_{\alpha,D-1}, Q_\alpha \in A_0$ are relatively prime, and $\alpha = Q_\alpha^{-1} \sum_{j=0}^{D-1} P_{\alpha,j} \omega^j$
$\overline{h}(\alpha)$ $(\alpha \in K)$	$\max(h(P_{\alpha,0}), \dots, h(P_{\alpha,D-1}), h(Q_\alpha))$

History and Summary

First, we give a brief historical overview of the equations treated in our book, and then outline the contents of the book.

We start with **ineffective** results. Thue (1909) developed an ingenious method for approximation of algebraic numbers by rationals. As an application, he proved that if $F \in \mathbb{Z}[X,Y]$ is a binary form (i.e., a homogeneous polynomial) of degree at least 3, which is irreducible over $\mathbb{Q}$ and δ is a nonzero integer, then the equation

$$F(x,y) = \delta \text{ in } x,y \in \mathbb{Z} \tag{1}$$

(nowadays called a *Thue equation*) has only finitely many solutions. Thue's approximation result was later considerably improved and generalized by many people including Siegel, Mahler, Dyson, Gel'fond, Roth, Schmidt, and Schlickewei.

Thue's finiteness theorem concerning equation (1) has many generalizations. Siegel (1921) generalized it for the number field case when the ground ring, i.e., the ring from which the solutions are taken, is the ring of integers $\mathcal{O}_K$ of a number field K. Mahler (1933) extended Thue's theorem to the case of ground rings of the form $\mathbb{Z}[(p_1,\ldots,p_s)^{-1}]$, where $p_1,\ldots,p_s$ are primes, while Parry (1950) gave a common generalization of the results of Siegel and Mahler to the case where the ground ring is the ring of S-integers of a number field.

Siegel's theorem has the following important consequence, which was not stated explicitly by Siegel but was implicitly proved by him. Denote by $\mathcal{O}_K^*$ the group of units of $\mathcal{O}_K$, and let α and β be nonzero elements of the number field K. Using the fact that $\mathcal{O}_K^*$ is finitely generated, it is easy to deduce from Siegel's theorem that the equation

$$\alpha x + \beta y = 1 \tag{2}$$

in $x, y \in \mathcal{O}_K^*$ has only finitely many solutions. Similarly, it follows from the results of Mahler and Parry that equation (2) has finitely many solutions even in S-units of K; these are elements of K composed of prime ideals from a finite, possibly empty set S of prime ideals of $\mathcal{O}_K$. Nowadays equation (2) is called a *unit equation* (when S is empty) resp. *S-unit equation* otherwise, or more precisely a unit equation and S-unit equation in two unknowns.

Further important equations are

$$f(x) = \delta y^m \text{ in } x, y \in \mathbb{Z}, \tag{3}$$

where $f \in \mathbb{Z}[X]$ is a polynomial of degree n and $\delta \in \mathbb{Z}\backslash\{0\}$. Equation (3) is called *elliptic* if $n = 3$ and $m = 2$, more generally *hyperelliptic* if $n \geq 3$ and $m = 2$, and *superelliptic* if $n \geq 2$ and $m \geq 3$. If m or n is at least 3 and f has no multiple zeros, equation (3) has only finitely many solutions. This was proved in the elliptic case by Mordell (1922a, 1922b, 1923), in the hyperelliptic case by Siegel (1929), and in the superelliptic case by Siegel (1929). LeVeque (1964) considered (3) in the more general case when f may have multiple zeros, and gave a finiteness criterion for (3) over the ring of integers of a number field.

A celebrated theorem of Siegel (1929) states that if $F(X,Y)$ is a polynomial with coefficients in a number field K, which is irreducible over $\overline{K}$, and the affine curve $F(x,y) = 0$ is of genus ≥ 1, then this curve has only finitely many points with integral coordinates in K. This theorem implies the above-mentioned finiteness results on Thue equations, unit equations, and hyperelliptic/superelliptic equations over number fields.

Lang (1960) generalized Siegel's theorem to what we call the **finitely generated** case, when the solutions are taken from an arbitrary integral domain of characteristic 0 that is finitely generated as a $\mathbb{Z}$-algebra, that is, a domain of the shape

$$\mathbb{Z}[z_1, \ldots, z_r] = \{f(z_1, \ldots, z_r) : f \in \mathbb{Z}[X_1, \ldots, X_r]\},$$

where $z_1, \ldots, z_r$ may be algebraic or transcendental over $\mathbb{Q}$. Recall that both the ring of integers of a number field K and the rings of S-integers of K are of this shape, with $z_1, \ldots, z_r$ all algebraic. In his proof, Lang used a specialization argument, reducing the theorem to the case of number fields and function fields of one variable, and then applied Siegel's theorem (1929) and its function field analogue from Lang (1960). As a consequence, Lang extended the earlier finiteness results concerning Thue equations, unit equations, and hyperelliptic/superelliptic equations to the finitely generated case.

Multivariate generalizations of Thue equations that have attracted much attention are the *decomposable form equations*

$$F(x_1, \ldots, x_m) = \delta \text{ in } x_1, \ldots, x_m \in \mathbb{Z}, \tag{4}$$

where $\delta \in \mathbb{Z}\backslash\{0\}$ and $F(X_1, \dots, X_m)$ is a decomposable form of degree $n > m$ in $m \geq 2$ variables with coefficients in $\mathbb{Z}$, i.e., a homogeneous polynomial, which factorizes into linear forms with coefficients in the algebraic closure $\overline{\mathbb{Q}}$. Further important types of decomposable form equations are *norm form equations*, *discriminant form equations*, and *index form equations*, which are of basic importance in algebraic number theory. Schmidt (1971, 1972) developed a multidimensional generalization of Roth's theorem on the approximation of algebraic numbers, eventually leading to his famous Subspace Theorem, and from the latter he deduced a finiteness criterion for norm form equations. Evertse and Győry (1988b) proved a general finiteness criterion for decomposable form equations of the form (4). Their proof depends on the following finiteness result on *multivariate unit equations* of the form

$$\alpha_1 x_1 + \cdots + \alpha_m x_m = 1 \text{ in } x_1, \dots, x_m \in \mathcal{O}_K^*, \tag{5}$$

where K is a number field and $\alpha_1, \dots, \alpha_m$ are nonzero elements of K. A solution of (5) is called *degenerate* if there is a vanishing subsum on the left hand side of (5). In this case (5) has infinitely many solutions if $\mathcal{O}_K^*$ is infinite. As a generalization of Siegel's theorem on equation (2), van der Poorten and Schlickewei (1982) and Evertse (1984) proved independently of each other that equation (5) has only finitely many non-degenerate solutions. This theorem was extended by Evertse and Győry (1988a) and van der Poorten and Schlickewei (1991) to the finitely generated case, when K is a finitely generated extension of $\mathbb{Q}$ and $\mathcal{O}_K^*$ is replaced by a finitely generated multiplicative subgroup of K^*. As a consequence, the above-mentioned general finiteness criterion for (4) was proved in Evertse and Győry (1988b) in a more general form, over finitely generated domains of characteristic 0.

In the 1960s, Baker developed an effective method in transcendence theory, providing nontrivial effective lower bounds for linear forms in logarithms of algebraic numbers. This furnished a very powerful tool to prove **effective** finiteness results for Diophantine equations over $\mathbb{Z}$ and more generally over number fields that enabled one to determine, at least in principle, all solutions of the equations under consideration. Using his method, Baker (1968b, 1968c, 1969) derived explicit upper bounds among others for the solutions of Thue equations and hyperelliptic/superelliptic equations. Győry (1974, 1979) used Baker's theory of logarithmic forms to obtain explicit upper bounds for the solutions of unit equations and S-unit equations in two unknowns (Evertse, Győry, Stewart and Tijdeman, 1988b). With the help of his bounds, Győry proved effective finiteness theorems for *discriminant equations* for polynomials

$$D(f) = \delta \text{ in monic polynomials } f \in \mathbb{Z}[X] \tag{6}$$

and for elements

$$D(\alpha) = \delta \text{ in algebraic integers } \alpha. \tag{7}$$

Here, $D(\ )$ denotes the discriminant of a polynomial f resp. of an algebraic integer α, and δ is a nonzero integer. Two monic polynomials $f, f' \in \mathbb{Z}[X]$ are called *strongly* $\mathbb{Z}$*-equivalent* if $f'(X) = f(X + a)$ for some $a \in \mathbb{Z}$. Similarly, two algebraic integers α and α' are said to be *strongly* $\mathbb{Z}$*-equivalent* if $\alpha' - \alpha \in \mathbb{Z}$. Clearly, strongly $\mathbb{Z}$-equivalent monic polynomials resp. algebraic integers have the same discriminant.

Győry (1973) proved that there are only finitely many pairwise strongly $\mathbb{Z}$-inequivalent monic polynomials with the property (6). A similar finiteness theorem was proved for the solutions of (7) by Birch and Merriman (1972), and independently by Győry (1973). Győry's proofs for (6) and (7) are effective. These results, in less precise form, were generalized in Győry (1978a) for the number field case, and in Győry (1982) in an ineffective form, for the finitely generated case, subject to the condition that the ground ring is integrally closed. These results have many applications, among others, to power integral bases of ring extensions.

By using Győry's bounds on the solutions of unit equations in two unknowns, Győry (1976, 1980a) and Győry and Papp (1978) generalized Baker's effective theorem on Thue equations to equations in arbitrarily many unknowns. They derived explicit bounds for the solutions of a class of decomposable form equations over number fields, including discriminant form equations and certain norm form equations.

Tijdeman (1976) used Baker's theory of logarithmic forms to give an explicit upper bound for the solutions of the *Catalan equation*

$$x^m - y^n = 1 \text{ in positive integers } x, y, m, n \text{ with } m, n > 1 \text{ and } mn > 4. \tag{8}$$

Further, when in equation (3) m is also unknown and f has at least two distinct zeros, Schinzel and Tijdeman (1976) gave an effective upper bound for m. In this case, equation (3) is now called the *Schinzel–Tijdeman equation*. It is interesting to note that the effective theorems of Tijdeman (1976) and Schinzel and Tijdeman (1976) had no previously ineffective versions.

For Thue equations, unit equations, and hyper/superelliptic equations, analogous effective results were obtained by Mason (1981, 1983, 1984) and others over function fields of characteristic 0. The above-mentioned effective results over number fields and function fields were later improved and generalized by many people, and led to several further applications.

In Győry (1983, 1984b), the author extended the effective finiteness theorems concerning Thue equations, discriminant equations, and a class of decomposable form equations over number fields to similar such equations over restricted classes of finitely generated domains of characteristic 0, which may contain both algebraic and transcendental elements. To prove these extensions, Győry developed an effective specialization method to reduce the general equations under consideration to equations of the same type over number fields and function fields, and then used effective results concerning these reduced equations to derive effective bounds for the solutions of the initial equations.

Evertse and Győry (2013) refined the method of Győry and proved effective finiteness theorems for unit equations in two unknowns in full generality, over arbitrary finitely generated domains of characteristic 0 over $\mathbb{Z}$. In fact, they obtained their results by combining Győry's techniques with the work of Aschenbrenner (2004) concerning the effective resolution of systems of linear equations over polynomial rings $\mathbb{Z}[X_1, \ldots, X_n]$.

The general effective specialization method of Evertse and Győry led to effective finiteness results over finitely generated domains for several other classes of Diophantine equations, such as Thue equations, hyper/superelliptic equations, and the Schinzel–Tijdeman equation (Bérczes, Evertse and Győry 2014), a generalization of unit equations (Bérczes, 2015a, 2015b), and the Catalan equation (Koymans 2016, 2017). Further, generalizing another method of Győry (1973) and Győry and Papp (1978) applied over number fields, the present authors in Evertse and Győry (2017a, 2017b) and in Sections 2.6 and 2.8 of this book obtained effective finiteness theorems for decomposable form equations and discriminant equations over finitely generated domains. This other method is not based on specialization but instead uses a reduction of the equation under consideration to unit equations in two unknowns.

It is important to note that with the exception of discriminant equations and hyper- and superelliptic equations, both methods mentioned above provide quantitative results over finitely generated domains, giving effective bounds for the solutions. This is due to the effective and quantitative feature of the main tools from Chapters 4 to 8.

Major open problems are to make effective the general finiteness theorems of Siegel (1929) on integral points of curves and of van der Poorten and Schlickewei (1982) and Evertse (1984) on multivariate unit equations over number fields. Such effective versions could be extended to the finitely generated case, using existing analogues over function fields and applying our general effective specialization method.

We now outline the contents of our book. In Chapter 1, we present the most general ineffective finiteness results over finitely generated domains for Thue equations, unit equations in two unknowns, a generalization of unit equations, hyper- and superelliptic equations, curves of genus ≥1 with finitely many integral points, decomposable form equations, multivariate unit equations, and discriminant equations. Further, except for curves of genus ≥1 and multivariate unit equations, we cite the most general effective versions concerning the equations mentioned over number fields.

In Chapter 2, we state general effective finiteness theorems over finitely generated domains of characteristic 0 for unit equations in two unknowns, Thue equations, hyper- and superelliptic equations, the Schinzel–Tijdeman equation, the Catalan equation, decomposable form equations, and discriminant equations. As was mentioned above, apart from discriminant equations, the other results give also effective bounds for the solutions.

Chapter 3 is devoted to a short explanation of our general effective methods.

In Chapters 4 and 5, those effective results are collected on the above equations over number fields and function fields that are needed in Chapters 9 and 10, in the proofs of the general effective theorems stated in Chapter 2. We have skipped the complete proofs of the theorems in Chapters 4 and 5, which are rather technical. Instead, we sketch the proofs in simplified forms, which give sufficient insight into the main ideas.

Chapters 6–8 contain further important tools. In Chapter 6, we have collected results from effective commutative algebra; in Chapter 7, we give the detailed treatment of our effective specialization method; and in Chapter 8, we prove some useful results for "degree-height estimates," which may be viewed as an analogue of the naive height estimates of algebraic numbers for elements of the algebraic closure of a finitely generated field.

Lastly, in Chapters 9 and 10, the results and methods from Chapters 4 to 8 are combined to prove the general effective results presented in Chapter 2.