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Ineffective Results for Diophantine Equations over Finitely Generated Domains

This book is about Diophantine equations where the solutions are taken from an integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$, that is, from a domain of the shape

$$\mathbb{Z}[z_1, \dots, z_r] = \{f(z_1, \dots, z_r) : f \in \mathbb{Z}[X_1, \dots, X_r]\}$$

whose quotient field is of characteristic 0. The generators $z_1, \dots, z_r$ may be either algebraic or transcendental over $\mathbb{Q}$.

For instance, let K be a number field and $\mathcal{O}_K$ its ring of integers. Let $\{\omega_1, \dots, \omega_d\}$ be a $\mathbb{Z}$-module basis of $\mathcal{O}_K$. Then $\mathcal{O}_K = \mathbb{Z}[\omega_1, \dots, \omega_d]$.

More generally, let K be a number field and with the notation introduced in Section 4.2, let S be a finite set of places of K, consisting of all infinite places of K and of the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ of $\mathcal{O}_K$. Then the ring of S-integers of K, denoted by $\mathcal{O}_S$, is given by the set of all elements α of K such that there are non-negative integers $k_1, \dots, k_t$ with $\alpha\mathfrak{p}_1^{k_1} \cdots \mathfrak{p}_t^{k_t} \subseteq \mathcal{O}_K$. In the particular case that S consists only of the infinite places of K, the ring $\mathcal{O}_S$ is just equal to $\mathcal{O}_K$. We may express $\mathcal{O}_S$ otherwise as

$$\mathcal{O}_S = \mathbb{Z}[\omega_1, \dots, \omega_d, \pi^{-1}],$$

where again, $\{\omega_1, \dots, \omega_d\}$ is a $\mathbb{Z}$-module basis of $\mathcal{O}_K$ and where $\pi\mathcal{O}_K = (\mathfrak{p}_1, \dots, \mathfrak{p}_t)^{h_K}$ with h_K being the class number of K. Thus, both the ring of integers and the rings of S-integers of a number field are domains finitely generated over $\mathbb{Z}$, with algebraic generators.

In general, we will consider Diophantine equations over integral domains $\mathbb{Z}[z_1, \dots, z_r]$ where some of the generators, say $z_1, \dots, z_q$, are algebraically independent of $\mathbb{Q}$, and the other generators are algebraic over $\mathbb{Q}(z_1, \dots, z_q)$.

In this chapter, we present the most important **ineffective** finiteness theorems for integral solutions of various classes of Diophantine equations, including *Thue equations, unit equations, hyper- and superelliptic equations, equations involving integral points on curves, decomposable form equations, and discriminant equations.* We consider these classes of equations in separate sections. For each class, we state the finiteness results in their most general form, over an arbitrary integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$, and give an account of the earlier special cases, leading to the general result. Over $\mathbb{Z}$ or more generally over the rings of integers or S-integers of number fields, these results were proved mostly by the powerful Thue–Siegel–Roth–Schmidt method, while in the finitely generated case, the equations are reduced either to the number field and function field cases by means of some specialization arguments or to such equations for which the finiteness of the number of solutions is already proved; see, e.g., Lang (1960), Győry (1982), Evertse and Győry (1988a, 1988b), and van der Poorten and Schlickewei (1991). At the end of each section, we make a mention to the corresponding **effective** results over $\mathbb{Z}$ or over number fields whose general versions over finitely generated domains will be presented in Chapter 2.

The above-mentioned equations have been studied very extensively, and they have many important generalizations, analogues, and applications. For details, we refer, e.g., to the books Lang (1962, 1978, 1983), Borevich and Shafarevich (1967), Mordell (1969), Baker (1975), Győry (1980b), Evertse (1983), Mason (1984), Shorey and Tijdeman (1986), Schmidt (1991), Sprindžuk (1993), Bombieri and Gubler (2006), Zannier (2009), Evertse and Győry (2015, 2017a), Bugeaud (2018), and the survey papers of Evertse, Győry, Stewart, and Tijdeman (1988b), and Győry (1984a, 1992, 2002).

1.1 Thue Equations

Let A denote an integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$. Let K denote the quotient field of A and fix an algebraic closure $\overline{K}$ of K. We first consider the equation

$$F(x,y) = \delta \quad \text{in} \quad x,y \in A \tag{1.1.1}$$

over A, where $F(X,Y)$ is a binary form of degree n with coefficients in A and $\delta \in A\setminus\{0\}$.

The following result is a consequence of the more general Theorem 1.4.1, which will be stated in Section 1.4.

Theorem 1.1.1 *Assume that F has at least three pairwise nonproportional linear factors over $\overline{K}$. Then equation* (1.1.1) *has only finitely many solutions.*

The condition in the theorem is obviously satisfied if F has degree at least 3 and its discriminant is nonzero. This theorem cannot be extended to binary forms F with fewer than three pairwise nonproportional linear factors; for instance, the Pell equation $x^2 - dy^2 = 1$ over $\mathbb{Z}$, where d is a positive integer not being a square, has infinitely many solutions.

In the classical case $A = \mathbb{Z}$, Theorem 1.1.1 was proved by Thue (1909). In fact, Thue proved it for irreducible F, but the general case can be easily reduced to the irreducible one. The proof of Thue's theorem is based on his result concerning approximations of algebraic numbers by rationals. After Thue, equations of the shape (1.1.1) are named *Thue equations*.

Thue's theorem has been generalized by many people. Siegel (1921) extended it to the case when A is the ring of integers of a number field, and Mahler (1933) extended it to rings of the shape $\mathbb{Z}[(p_1,\dots,p_s)^{-1}]$, where $p_1,\dots,p_s$ are distinct primes. Parry (1950) gave a common generalization of the results of Siegel and Mahler to rings of S-integers of a number field. In the above general form, Theorem 1.1.1 is due to Lang (1960).

We would like to mention another equivalent formulation of Theorem 1.1.1. First, we recall a result of Mahler (1933). Let $F \in \mathbb{Z}[X,Y]$ be a binary form with at least three pairwise nonproportional linear factors over $\overline{\mathbb{Q}}$, and let $p_1,\dots,p_s$ be distinct prime numbers. Then the equation

$$F(x,y) = \pm p_1^{z_1}\cdots p_s^{z_s} \text{ in } x,y,z_1,\dots,z_s \in \mathbb{Z} \text{ with } \gcd(x,y)=1 \tag{1.1.2}$$

has only finitely many solutions. If we drop the restriction $\gcd(x,y) = 1$, then we can construct infinite classes of solutions by multiplying (x,y) with products of powers of $p_1,\dots,p_s$. Thus, it is easily seen that Mahler's result can be translated as follows. Let $S = \{p_1,\dots,p_s\}$ be a finite set of primes, $\mathbb{Z}_S = \mathbb{Z}[(p_1,\dots,p_s)^{-1}]$ the corresponding ring of S-integers, and $\mathbb{Z}_S^* = \{\pm p_1^{z_1}\cdots p_s^{z_s} : z_1,\dots,z_s \in \mathbb{Z}\}$ the group of units of $\mathbb{Z}_S$. Then the solutions of

$$F(x,y) \in \mathbb{Z}_S^* \text{ in } (x,y) \in \mathbb{Z}_S^2 \tag{1.1.3}$$

lie in finitely many $\mathbb{Z}_S^*$-cosets, where a $\mathbb{Z}_S^*$-coset is a set of solutions of the shape $\{u\cdot(x_0,y_0) : u \in \mathbb{Z}_S^*\}$, with $(x_0,y_0) \in \mathbb{Z}_S^2$ fixed.

We now generalize this last equation to arbitrary finitely generated domains of characteristic 0 that are finitely generated over $\mathbb{Z}$. Let A be such a domain and denote by A^* its unit group, i.e., the group of invertible elements. Further, let $F \in A[X,Y]$ be a binary form and δ a nonzero element of A, and consider the following generalization of (1.1.3):

$$F(x,y) \in \delta A^* \text{ in } (x,y) \in A^2. \tag{1.1.4}$$

Because of its connection with (1.1.2), equation (1.1.4) is called a *Thue–Mahler equation*. Just like above, we can divide the solutions $(x, y) \in A^2$ of (1.1.4) into A^*-*cosets* $A^*(x_0, y_0) = \{u \cdot (x_0, y_0) : u \in A^*\}$.

The following assertion is equivalent to Theorem 1.1.1.

Theorem 1.1.2 *Assume again that F has at least three pairwise nonproportional linear factors over $\overline{K}$. Then equation* (1.1.4) *has only finitely many A^*-cosets of solutions.*

Theorem 1.1.1⇒Theorem 1.1.2 Assume Theorem 1.1.1. According to a theorem of Roquette (1957), the unit group A^* is finitely generated. Let $\{v_1, \ldots, v_s\}$ be a set of generators for A^*, and define $\mathcal{U} := \{v_1^{m_1} \cdots v_s^{m_s} : m_1, \ldots, m_s \in \{0, \ldots, n-1\}\}$. Then every element of A^* can be expressed as $u_1 u_2^n$, where $u_1 \in \mathcal{U}$ and $u_2 \in A^*$. Clearly, if $(x, y) \in A^2$ satisfies (1.1.4), then $F(x, y) = \delta u_1 u_2^n$ for some $u_1 \in \mathcal{U}$, $u_2 \in A^*$, and so $F(x', y') = \delta u_1$, where $(x', y') = u_2^{-1}(x, y)$. Hence, every A^*-coset of solutions of (1.1.4) contains (x', y') with $F(x', y') = \delta u_1$ with some $u_1 \in \mathcal{U}$, and Theorem 1.1.1 implies that for each $u_1 \in \mathcal{U}$, there are only finitely many possibilities for (x', y'). This implies Theorem 1.1.2.

Theorem 1.1.2⇒Theorem 1.1.1 Assume Theorem 1.1.2. Let $A^*(x_0, y_0)$ be one of the finitely many A^*-cosets of solutions of (1.1.4) and pick those solutions from it that satisfy (1.1.1). These solutions are all of the shape $u(x_0, y_0)$ with $u^n = F(x_0, y_0)/\delta$, and there are only finitely many of those. Hence, (1.1.1) has only finitely many solutions. □

Equation (1.1.1) has many further generalizations, see, e.g., equation (1.4.1) in Section 1.4, equations (1.5.1), (1.5.2), and (1.5.4) in Section 1.5, and Evertse and Győry (2015, Chapter 9).

In the case $A = \mathbb{Z}$, the first general **effective** result for equation (1.1.1) was established by Baker (1968b). He gave an explicit upper bound for the solutions by means of his effective method based on lower bounds for linear forms in logarithms. Coates (1969) extended Baker's result to the case of ground rings of the type $A = \mathbb{Z}[(p_1 \cdots p_s)^{-1}]$, and later, Kotov and Sprindžuk (1973) extended that to the case when A is the ring of S-integers of a number field. Győry (1983), using his effective specialization method, generalized the above results for a wide but special class of finitely generated domains that may contain both algebraic and transcendental elements. In Chapter 2, Theorem 2.3.1 gives an effective version of Theorem 1.1.1 in quantitative form over an arbitrary integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$. Its proof uses a precise effective version of Theorem 1.1.1 over rings of S-integers of number fields; see Theorem 4.4.1 in Chapter 4, as well as an effective version over function fields, see Theorem 5.4.1 in Chapter 5, which is a slight variation of a result of Mason (1981, 1984).

1.2 Unit Equations in Two Unknowns

Let again A be an integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$ and K its quotient field. Further, let a and b be the nonzero elements of K. Consider the *unit equation*

$$ax + by = 1 \quad \text{in} \quad x, y \in A^*, \tag{1.2.1}$$

where A^* denotes the unit group of A, i.e., the multiplicative group of invertible elements of A.

By a theorem of Roquette (1957), the group A^* is finitely generated. Lang (1960) proved the following general result.

Theorem 1.2.1 *Equation* (1.2.1) *has only finitely many solutions.*

The first finiteness result for equation (1.2.1) was implicitly proved by Siegel (1921) in the case where K is a number field and A is the ring of integers of K. For the case when A is of the type $\mathbb{Z}[(p_1 \cdots p_s)^{-1}]$ with distinct primes $p_1, \dots, p_s$, the finiteness of the number of solutions was obtained by Mahler (1933), while a common generalization of the results of Siegel and Mahler follows from Parry (1950).

In fact, in Lang (1960), the following more general version of Theorem 1.2.1 is established.

Theorem 1.2.2 *Let K be a field of characteristic* 0*, a and b the nonzero elements of K, and Γ a finitely generated multiplicative subgroup of K^*. Then the equation*

$$ax + by = 1 \quad \text{in} \quad x, y \in \Gamma \tag{1.2.2}$$

has only finitely many solutions.

Proof Using an argument due to Siegel (1921), the theorem can be easily reduced to Theorem 1.1.1. Indeed, suppose that equation (1.2.2) has infinitely many solutions. Let n be an integer ≥ 3. Since Γ is finitely generated, the quotient group Γ/Γ^n is finite. Hence, there is a solution (x_0, y_0) of (1.2.2) such that there are infinitely many solutions x, y such that $x \in x_0\Gamma^n$ and $y \in y_0\Gamma^n$. Each of these solutions x, y can be written in the forms $x = x_0u^n$ and $y = y_0v^n$ with some $u, v \in \Gamma$. Denoting by A the ring generated by Γ over $\mathbb{Z}$, it follows that the Thue equation

$$(ax_0)u^n + (by_0)v^n = 1$$

has infinitely many solutions $u, v \in A$. This contradicts Theorem 1.1.1. □

We note that, conversely, Thue equations can be reduced to finitely many appropriate unit equations; see, e.g., Evertse and Győry (2015). In other words, Thue equations and unit equations in two unknowns are, in fact, equivalent. This was (implicitly) pointed out by Siegel (1926).

Theorem 1.2.2 has several generalizations, see, e.g., Theorem 1.5.4 in Section 1.5, Lang (1960, 1983), and Evertse and Győry (2015). Here we present one of them.

Lang (1960) extended his result concerning equation (1.2.2) to equations of the shape

$$F(x,y) = 0 \quad \text{in} \quad x, y \in \Gamma, \tag{1.2.3}$$

where Γ is again a finitely generated multiplicative subgroup of K, and where $F \in A[X,Y]$ is a nonconstant polynomial. He proved the following.

Theorem 1.2.3 *Let $F \in A[X,Y]$ be a nonconstant polynomial that is not divisible by any polynomial of the shape*

$$X^mY^n - \alpha \quad \text{or} \quad X^m - \alpha Y^n, \tag{1.2.4}$$

with $\alpha \in \Gamma$ and with non-negative integers m and n, not both zero. Then equation (1.2.3) *has only finitely many solutions.*

It is easy to see that the exceptions described in Theorem 1.2.3 must be excluded.

Lang (1965a, 1965b) conjectured that Theorem 1.2.3 remains valid if one replaces Γ by its division group $\overline{\Gamma}$, which consists of those $\gamma \in \overline{K}^*$ such that $\gamma^k \in \Gamma$ for some positive integer k. Hence, in this case, the solutions x, y do not necessarily belong to K. Lang's conjecture has been proved by Liardet (1974, 1975) who obtained the following.

Theorem 1.2.4 *Let $F \in A[X,Y]$ be a nonconstant polynomial that is not divisible by any polynomial of the shape* (1.2.4) *with $\alpha \in \overline{\Gamma}$ and with non-negative integers m and n, not both zero. Then equation* (1.2.3) *has only finitely many solutions even in $x, y \in \overline{\Gamma}$.*

The first general **effective** results for equation (1.2.1) over the ring of integers of algebraic number fields were proved in Győry (1972, 1973, 1974, 1976), over rings of S-integers of an algebraic number field in Győry (1979), and independently, in a less precise form, in Kotov and Trelina (1979). Using Baker's method concerning linear forms in logarithms, effective upper bounds were given for the solutions. These bounds were improved later by several authors; see, e.g., Bugeaud and Győry (1996a), Győry and Yu (2006), and Győry (2019).

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Over algebraic number fields, Bombieri and Gubler (2006) gave an effective version of Lang's theorem on equation (1.2.3), which was made explicit by Bérczes, Evertse, Győry, and Pontreau (2009). These results are proved under a slightly stronger condition than (1.2.4), with $\alpha \in \overline{K}$ used in place of $\alpha \in \Gamma$.

In the number field case, an effective version of Liardet's theorem for linear polynomials F is due to Bérczes, Evertse, and Győry (2009), and for the general case to Bérczes, Evertse, Győry, and Pontreau (2009).

In Section 2.2, we present effective versions of Theorems 1.2.1 and 1.2.2 in quantitative form over an arbitrary integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$; see Theorems 2.2.1 and 2.2.3. In its proof, we use the result of Győry and Yu (2006) concerning equation (1.2.1) for the group of S-units of a number field, as well as the Mason–Stothers abc-theorem for function fields (as in Mason (1984)), see Theorem 5.2.2 in Chapter 5. Further, we formulate some effective generalizations for equation (1.2.3), due to Bérczes (2015a, 2015b), see Theorems 2.2.4 and 2.2.5.

1.3 Hyper- and Superelliptic Equations

Now consider the equation

$$f(x) = \delta y^m \quad \text{in} \quad x, y \in A, \tag{1.3.1}$$

where A is again an integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$, $f \in A[X]$ is a polynomial of degree $n \geq 2$, $\delta \in A\backslash\{0\}$, and $m \geq 2$ is an integer. Equation (1.3.1) is called *elliptic* if $n = 3$ and $m = 2$, *hyperelliptic* if $n \geq 3$ and $m = 2$, and *superelliptic* if $n \geq 2$ and $m \geq 3$.

The following theorem follows from the general ineffective Theorem 1.4.1 of Lang.

Theorem 1.3.1 *Suppose that in* (1.3.1) *m or n is at least* 3 *and that f has no multiple zeros. Then* (1.3.1) *has only finitely many solutions.*

Under the assumptions of Theorem 1.3.1, the affine curve $f(x) - \delta y^m = 0$ has genus ≥ 1. Thus, Theorem 1.3.1 is a consequence of the general Theorem 1.4.1 stated below on the finiteness of the number of intregral points on algebraic curves. The example of Pell equations shows that (1.3.1) may have infinitely many solutions if $m = 2$ and $n = 2$.

In the special case $A = \mathbb{Z}$, Mordell (1922a, 1922b, 1923) proved the finiteness of the numbers of solutions of elliptic equations for which the polynomial f has no multiple zeros. In particular, this implies that for every nonzero integer k, the *Mordell equation* $x^3 + k = y^2$ has only finitely many solutions. Mordell's

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finiteness results were extended by Siegel (1926) to hyperelliptic equations, by reducing such equations to unit equations. LeVeque (1964) considered (1.3.1), where f may have multiple zeros, and gave a finiteness criterion for the equation (1.3.1) when A is the ring of integers of a number field. The proofs of Mordell, Siegel, and LeVeque are ineffective.

Over $\mathbb{Z}$, Baker (1968b, 1968c, 1969) was the first to give **effective** upper bounds for the solutions of (1.3.1) in the case when f has at least three simple zeros if $m = 2$ and at least two simple zeros if $m \geq 3$. Brindza (1984) made LeVeque's theorem effective and extended it to S-integral solutions from a number field.

Schinzel and Tijdeman (1976) considered equation (1.3.1) in the more general situation when m is also unknown. In the case that $A = \mathbb{Z}$ and that f has at least two distinct zeros, they derived an effective upper bound for m. Equation (1.3.1) with m also unknown is nowadays called the *Schinzel–Tijdeman equation*. All the effective results mentioned above depend on Baker's method.

In Chapter 2, we present effective versions of Theorem 1.3.1 and the Schinzel–Tijdeman theorem in quantitative form, over an arbitrary integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$; see Theorems 2.4.1 and 2.4.2. These results follow from similar effective results over number fields (see Theorems 4.5.1–4.5.3) and function fields (see Theorems 5.5.1 and 5.5.2).

1.4 Curves with Finitely Many Integral Points

Let K be a finitely generated extension of $\mathbb{Q}$ and A a subring of K that is finitely generated over $\mathbb{Z}$. The following finiteness theorem is of fundamental importance in Diophantine number theory.

Theorem 1.4.1 *Let $F \in K[X,Y]$ be a polynomial irreducible over $\overline{K}$ such that the affine curve $F(x,y) = 0$ is of genus ≥ 1. Then this curve has only finitely many points with coordinates in A.*

In other words, under the above assumptions, the equation

$$F(x,y) = 0 \quad \text{in} \quad x,y \in A \tag{1.4.1}$$

has only finitely many solutions.

In the case when K is a number field and A its ring of integers, this celebrated theorem was proved by Siegel (1929). Further, Siegel described the cases when the curve has genus 0 and has infinitely many points with coordinates in A. Mahler (1934) conjectured that a similar statement holds for rational points with coordinates having only finitely many fixed primes in their denominators, and proved this for curves of genus 1. In the above general form, Theorem 1.4.1 is due to Lang (1960); see also Lang (1962, 1983). In this proof, Lang used a specialization argument, reducing Theorem 1.4.1 to the case of number fields resp. function fields of one variable, and then applied Siegel's theorem and its analogue over function fields from Lang (1960).

Confirming Mordell's (1922a) famous conjecture on rational points on curves, Faltings (1983) proved, first for number fields K and later for finitely generated extensions K of $\mathbb{Q}$; (see Faltings and Wüstholz [1984, p. 205, Theorem 3]), that if the above curve has genus ≥ 2, then it has only finitely many points even with coordinates in K as well. Except for the genus 1 case, Faltings' theorem contains Theorem 1.4.1.

All known proofs of Theorem 1.4.1 and those of Faltings are ineffective. As was mentioned in Sections 1.1–1.3, Theorem 1.4.1 has been made **effective** in a couple of important special cases. Further, in the case when K is a number field, an effective version of Theorem 1.4.1 for genus 1 curves was obtained by Baker and Coates (1970).

It is a **major open problem** to give an effective version of Theorem 1.4.1 in full generality.

1.5 Decomposable Form Equations and Multivariate Unit Equations

Let K be a finitely generated extension field of $\mathbb{Q}$ and $F \in K[X_1, \dots, X_m]$ a decomposable form in $m \geq 2$ variables, i.e., F factorizes into linear forms over an extension of K, which we may choose to be a given algebraic closure $\overline{K}$ of K. Let $\delta \in K^*$ and A be a subring of K that is finitely generated over $\mathbb{Z}$. As a generalization of the Thue equation, we consider the *decomposable form equation*

$$F(\mathbf{x}) = \delta \ \text{ in } \mathbf{x} = (x_1, \dots, x_m) \in A^m. \tag{1.5.1}$$

Let $\mathcal{L}_0$ be a maximal set of pairwise linearly independent linear factors of F. That is, we can express F as $c\ell_1^{e_1} \cdots \ell_n^{e_n}$, where $\mathcal{L}_0 = \{\ell_1, \dots, \ell_n\}$, $c \in K^*$, and $e_1, \dots, e_n$ are positive integers. For applications, it is convenient to consider the following generalization of equation (1.5.1). Let $\mathcal{L} \supseteq \mathcal{L}_0$ be a finite set of pairwise linearly independent linear forms of $X_1, \dots, X_m$, with coefficients in $\overline{K}$, and consider now the equation

$$F(\mathbf{x}) = \delta \ \text{ in } \mathbf{x} = (x_1, \dots, x_m) \in A^m \text{ with } \ell(\mathbf{x}) \neq 0 \text{ for all } \ell \in \mathcal{L}. \tag{1.5.1a}$$

For $\mathcal{L} = \mathcal{L}_0$, equation (1.5.1a) gives (1.5.1).

To state the main results, we need some definitions. Given a nonzero linear subspace V of the K-vector space K^m and linear forms $\ell_1, \ldots, \ell_r$ in $\overline{K}[X_1, \ldots, X_m]$, we say that $\ell_1, \ldots, \ell_r$ are linearly dependent on V if there are $c_1, \ldots, c_r \in \overline{K}$, not all 0, such that $c_1\ell_1 + \cdots + c_r\ell_r$ vanishes identically on V. Otherwise, we say that $\ell_1, \ldots, \ell_r$ are linearly independent on V.

We say that a nonzero linear subspace V of K^m is *$\mathcal{L}$-nondegenerate* if $\mathcal{L}$ contains $r \geq 3$ linear forms $\ell_1, \ldots, \ell_r$ that are linearly dependent on V, while each pair ℓ_i, ℓ_j $(i \neq j)$ is linearly independent of V. Otherwise, the space V is called *$\mathcal{L}$-degenerate*. That is, V is $\mathcal{L}$-degenerate precisely if there are $\ell_1, \ldots, \ell_r \in \mathcal{L}$ such that $\ell_1, \ldots, \ell_r$ are linearly independent of V, while each other's linear form $\ell \in \mathcal{L}$ is linearly dependent on V to one of $\ell_1, \ldots, \ell_r$. In particular, V is $\mathcal{L}$-degenerate if V has dimension 1.

Lastly, we call V *$\mathcal{L}$-admissible* if no linear form in $\mathcal{L}$ vanishes identically on V.

The following general finiteness criterion was proved by Evertse and Győry (1988b).

Theorem 1.5.1 *The following two statements are equivalent:*

(i) Every $\mathcal{L}$-admissible linear subspace of K^m of dimension ≥ 2 is $\mathcal{L}_0$-nondegenerate;

(ii) For every subring A of K that is finitely generated over $\mathbb{Z}$ and for every $\delta \in K^*$, equation (1.5.1a) *has only finitely many solutions.*

For $\mathcal{L} = \mathcal{L}_0$, this theorem gives a finiteness criterion for equation (1.5.1). It relates a statement (cf. (ii)) about the finiteness of the number of solutions to a condition (cf. (i)) that can be formulated in terms of linear algebra. It can be shown that (i) is effectively decidable once K, $\mathcal{L}_0$, and $\mathcal{L}$ are given in some explicit form; see Evertse and Győry (2015, Theorem 9.1.1) for an equivalent formulation of (i) for which the effective decidability is clear.

In the case $m = 2$ and $\mathcal{L} = \mathcal{L}_0$, Theorem 1.5.1 gives immediately Theorem 1.1.1 on Thue equations. For a more general version of Theorem 1.5.1, see Evertse and Győry (2015, Chapter 9).

Decomposable form equations are of basic importance in Diophantine number theory. Besides Thue equations (when $m = 2$), important classes of decomposable form equations are norm form equations, discriminant form equations, and index form equations.

Let us start with norm form equations. Let $\alpha_1 = 1, \alpha_2, \ldots, \alpha_m \in \overline{K}$ and suppose they are linearly independent over K. Put $K' := K(\alpha_1, \ldots, \alpha_m)$. Assume that K' is of degree $n \geq 3$ over K. Put $\ell(\mathbf{X}) := \alpha_1 X_1 + \cdots + \alpha_m X_m$ and denote by $\ell^{(i)}(\mathbf{X}) := \alpha_1^{(i)} X_1 + \cdots + \alpha_m^{(i)} X_m$ $(i = 1, \ldots, n)$ the conjugates of $\ell(\mathbf{X})$ with respect to K'/K. Then