The William Lowell Putnam Mathematical Competition 1985–2000

Problems, Solutions, and Commentary

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Dedicated to the Putnam contestants

Introduction

This book is the third collection of William Lowell Putnam Mathematical Competition problems and solutions, following [PutnamI] and [PutnamII]. As the subtitle indicates, the goals of our volume differ somewhat from those of the earlier volumes.

Many grand ideas of mathematics are best first understood through simple problems, with the inessential details stripped away. When developing new theory, research mathematicians often turn to toy† problems as a means of getting a foothold. For this reason, Putnam problems and solutions should be considered not in isolation, but instead in the context of important mathematical themes. Many of the best problems contain kernels of sophisticated ideas, or are connected to some of the most important research done today. We have tried to emphasize the organic nature of mathematics, by highlighting the connections of problems and solutions to other problems, to the curriculum, and to more advanced topics. A quick glance at the index will make clear the wide range of powerful ideas connected to these problems. For example, Putnam problems connect to the Generalized Riemann Hypothesis (1988B1) and the Weil Conjectures (1991B5 and 1998B6).

1 Structure of this book

The first section contains the problems, as they originally appeared in the competition, but annotated to clarify occasional infelicities of wording. We have included a list of the Questions Committee with each competition, and we note here that in addition Loren Larson has served as an *ex officio* member of the committee for nearly the entire period covered by this book. Next is a section containing a brief hint for each problem. The hints may often be more mystifying than enlightening. Nonetheless, we hope that they encourage readers to spend more time wrestling with a problem before turning to the solution section.

The heart of this book is in the solutions. For each problem, we include every solution we know, eliminating solutions only if they are essentially equivalent to one already given, or clearly inferior to one already given. Putnam problems are usually constructed so that they admit a solution involving nothing more than calculus, linear algebra, and a bit of real analysis and abstract algebra; hence we always

† A "toy" problem does not necessarily mean an easy problem. Rather, it means a relatively tractable problem where a key issue has been isolated, and all extraneous detail has been stripped away.

include one solution requiring no more background than this. On the other hand, as mentioned above, the problems often relate to deep and beautiful mathematical ideas, and concealing these ideas makes some solutions look like isolated tricks; therefore where germane we mention additional problems solvable by similar methods, alternate solutions possibly involving more advanced concepts, and further remarks relating the problem to the mathematical literature. Our alternate solutions are sometimes more terse than the first one. The top of each solution includes the score distribution of the top contestants: see page 51. When we write "see 1997A6," we mean "see the solution(s) to 1997A6 and the surrounding material."

After the solutions comes a list of the winning individuals and teams. This includes one-line summaries of the winners' histories, when known to us. Finally, we reprint an article by Joseph A. Gallian, "Putnam Trivia for the Nineties," and an article by Bruce Reznick, "Some Thoughts on Writing for the Putnam."

2 The Putnam Competition over the years

The competition literature states: "The competition began in 1938, and was designed to stimulate a healthy rivalry in mathematical studies in the colleges and universities of the United States and Canada. It exists because Mr. William Lowell Putnam had a profound conviction in the value of organized team competition in regular college studies. Mr. Putnam, a member of the Harvard class of 1882, wrote an article for the December 1921 issue of the *Harvard Graduates' Magazine* in which he described the merits of an intercollegiate competition. To establish such a competition, his widow, Elizabeth Lowell Putnam, in 1927 created a trust fund known as the William Lowell Putnam Intercollegiate Memorial Fund. The first competition supported by this fund was in the field of English and a few years later a second experimental competition was held, this time in mathematics between two institutions. It was not until after Mrs. Putnam's death in 1935 that the examination assumed its present form and was placed under the administration of the Mathematical Association of America."

Since 1962, the competition has consisted of twelve problems, usually numbered A1 through A6 and B1 through B6, given in two sessions of three hours each on the first Saturday in December. For more information about the history of the Putnam Competition, see the articles of Garrett Birkhoff and L. E. Bush in [PutnamI].

The competition is open to regularly enrolled undergraduates in the U.S. and Canada who have not yet received a college degree. No individual may participate in the competition more than four times. Each college or university with at least three participants names a team of three individuals. But the team must be chosen *before* the competition, so schools often fail to select their highest three scores; indeed, some schools are notorious for this. Also, the team rank is determined by the sum of the ranks of the team members, so one team member having a bad day can greatly lower the team rank. These two factors add an element of uncertainty to the team competition.

Prizes are awarded to the mathematics departments of the institutions with the five winning teams, and to the team members. The five highest ranking individuals are designated Putnam Fellows; prizes are awarded to these individuals and to each of the next twenty highest ranking contestants. One of the Putnam Fellows is also awarded the William Lowell Putnam Prize Scholarship at Harvard. Also, in some years, beginning in 1992, the Elizabeth Lowell Putnam Prize has been awarded to a woman whose performance has been deemed particularly meritorious. The winners of this prize are listed in the "Individual Results" section. The purpose of the Putnam Competition is not only to select a handful of prize winners, however; it is also to provide a stimulating challenge to all the contestants.

The nature of the problems has evolved. A few of the changes reflect changing emphases in the discipline of mathematics itself: for example, there are no more problems on Newtonian mechanics, and the number of problems involving extended algebraic manipulations has decreased. Other changes seem more stylistic: problems from recent decades often admit relatively short solutions, and are never open-ended.

The career paths of recent Putnam winners promise to differ in some ways from those of their predecessors recorded in [PutnamI]. Although it is hard to discern patterns among recent winners since many are still in school, it seems that fewer are becoming pure mathematicians than in the past. Most still pursue a Ph.D. in mathematics or some other science, but many then go into finance or cryptography, or begin other technology-related careers. It is also true that some earlier winners have switched from pure mathematics to other fields. For instance, David Mumford, a Putnam Fellow in 1955 and 1956 who later won a Fields Medal for his work in algebraic geometry, has been working in computer vision since the 1980s.

3 Advice to the student reader

The first lesson of the Putnam is: don't be intimidated. Some of the problems relate to complex mathematical ideas, but all can be solved using only the topics in a typical undergraduate mathematics curriculum, admittedly combined in clever ways. By working on these problems and afterwards studying their solutions, you will gain insight into beautiful aspects of mathematics beyond what you may have seen before.

Be patient when working on a problem. Learning comes more from struggling with problems than from solving them. If after some time, you are still stuck on a problem, see if the hint will help, and sleep on it before giving up. Most students, when they first encounter Putnam problems, do not solve more than a few, if any at all, because they give up too quickly. Also keep in mind that problem-solving becomes easier with experience; it is not a function of cleverness alone.

Be patient with the solutions as well. Mathematics is meant to be read slowly and carefully. If there are some steps in a solution that you do not follow, try discussing it with a knowledgeable friend or instructor. Most research mathematicians do the same when they are stuck (which is most of the time); the best mathematics research is almost never done in isolation, but rather in dialogue with other mathematicians, and in consultation of their publications. When you read the solutions, you will often find interesting side remarks and related problems to think about, as well as connections to other beautiful parts of mathematics, both elementary and advanced. Maybe you will create new problems that are not in this book. We hope that you follow up on the ideas that interest you most.

Year	Cut-off score for			
	Median	Top ~ 200	Honorable Mention	Putnam Fellow
1985	2	37	66	91
1986	19	33	51	81
1987	1	26	49	88
1988	16	40	65	110
1989	0	29	50	77
1990	2	28	50	77
1991	11	40	62	93
1992	2	32	53	92
1993	10	29	41	60
1994	3	28	47	87
1995	8	35	52	85
1996	3	26	43	76
1997	1	25	42	69
1998	10	42	69	98
1999	0	21	45	69
2000	0	21	43	90

TABLE 1. Score cut-offs

4 Scoring

Scores in the competition tend to be very low. The questions are difficult and the grading is strict, with little partial credit awarded. Students who solve one question and write it up perfectly do better than those with partial ideas for a number of problems.

Each of the twelve problems is graded on a basis of 0 to 10 points, so the maximum possible score is 120. Table 1 shows the scores required in each of the years covered in this volume to reach the median, the top 200, Honorable Mention, and the rank of Putnam Fellow (top five, or sometimes six in case of a tie). Keep in mind that the contestants are self-selected from among the brightest in two countries. As you can see from Table 1, solving a single problem should be considered a success. In particular, the Putnam is not a "test" with passing and failing grades; instead it is an open-ended challenge, a competition between you and the problems.

Along with each solution in this book, we include the score distribution of the top 200 or so contestants on that problem: see page 51. This may be used as a rough indicator of the difficulty of a problem, but of course, different individuals may find different problems difficult, depending on background. The problems with highest scores were 1988A1 and 1988B1, and the problems with the lowest scores were 1999B4 and 1999B5. When an easier problem was accidentally placed toward the end of the competition, the scores tended to be surprisingly low. We suspect that this is because contestants expected the problem to be more difficult than it actually was.

5 Some basic notation

The following definitions are standard in modern mathematics, so we use them throughout this book:

$$\mathbb{Z} = \text{the ring of integers} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$
$$\mathbb{Q} = \text{the field of rational numbers} = \{\, m/n : m, n \in \mathbb{Z}, n \neq 0 \,\}$$
$$\mathbb{R} = \text{the field of real numbers}$$
$$\mathbb{C} = \text{the field of complex numbers} = \{\, a + bi : a, b \in \mathbb{R} \,\}, \text{ where } i = \sqrt{-1}$$
$$\mathbb{F}_q = \text{the finite field of } q \text{ elements.}$$

The cardinality of a set S is denoted $\#S$ or sometimes $|S|$. If $a, b \in \mathbb{Z}$, then "$a \mid b$" means that a divides b, that is, that there exists $k \in \mathbb{Z}$ such that $b = ka$. Similarly, "$a \nmid b$" means that a does not divide b. The set of positive real numbers is denoted by $\mathbb{R}^+$.

We use the notation $\ln x$ for the natural logarithm function, even though in higher mathematics the synonym $\log x$ is more frequently used. It is tacitly assumed that the base of the logarithm, if unspecified, equals $e = 2.71828\dots$. If logarithms to the base 10 are intended, it is better to write $\log_{10} x$. More generally, $\log_a x = (\log x)/(\log a)$ denotes logarithm to the base a. In computer science, the notation $\lg n$ is sometimes used as an abbreviation for $\log_2 n$. (In number theory, when p is a prime number, $\log_p x$ sometimes also denotes the p-adic logarithm function [Kob, p. 87], a function with similar properties but defined on nonzero p-adic numbers instead of positive real numbers. But this book will have no need for this p-adic function.)

Rings for us are associative and have a multiplicative unit 1. If R is a ring, then $R[x]$ denotes the ring of all polynomials

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is any nonnegative integer, and $a_0, a_1, \dots, a_n \in R$. Also, $R[[x]]$ denotes the ring of formal power series

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

where the a_i belong to R.

If R is a ring and $n \geq 1$, $M_n(R)$ denotes the set of $n \times n$ matrices with coefficients in R, and $\mathrm{GL}_n(R)$ denotes the subset of matrices $A \in M_n(R)$ that have an inverse in $M_n(R)$. When R is a field, a matrix $A \in M_n(R)$ has such an inverse if and only if its determinant $\det(A)$ is nonzero; more generally, for any commutative ring, A has such an inverse if and only if $\det(A)$ is a unit of R. (The reason to insist that the determinant be a unit, and not just nonzero, is that it makes $\mathrm{GL}_n(R)$ a group under multiplication.) For instance, $\mathrm{GL}_2(\mathbb{Z})$ is the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$.

6 Acknowledgements

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`http://www.unl.edu/amc`

hosted by American Mathematics Competitions; hopefully these postings will continue in future years. We thank Gabriel Carroll, Sabin Cautis, Keith Conrad, Ioana Dumitriu, J.P. Grossman, Doug Jungreis, Andrew Kresch, Abhinav Kumar, Greg Kuperberg, Russ Mann, Lenny Ng, Naoki Sato, Dave Savitt, Hoeteck Wee, and Eric Wepsic, who read parts of this book and contributed many suggestions and ideas that were incorporated into the text, often without attribution. We thank Jerry Alexanderson, Loren Larson, and Roger Nelsen for detailed and helpful comments on the entire manuscript. We thank Pramod Achar, Art Benjamin, George Bergman, Mira Bernstein, Anders Buch, Robert Burckel, Ernie Croot, Charles Fefferman, Donald Sarason, Jun Song, Bernd Sturmfels, Mark van Raamsdonk, and Balint Virag for additional comments, and for suggesting references. We thank Joe Gallian and Bruce Reznick for permission to reprint their articles [G2] and [Re4].

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