

Problems

The Forty-Sixth William Lowell Putnam Mathematical Competition December 7, 1985

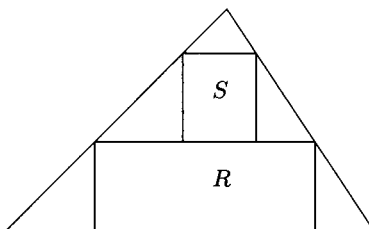
Questions Committee: Bruce Reznick, Richard P. Stanley, and Harold M. Stark
See page 35 for hints.

A1. Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

- (i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and
(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$,

where \emptyset denotes the empty set. Express the answer in the form $2^a 3^b 5^c 7^d$, where a, b, c , and d are nonnegative integers. (page 53)

A2. Let T be an acute triangle. Inscribe a pair R, S of rectangles in T as shown:



Let $A(X)$ denote the area of polygon X . Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above. (page 54)

A3. Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \dots$ by the condition

$$a_m(0) = d/2^m, \quad \text{and} \quad a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.$$

Evaluate $\lim_{n \rightarrow \infty} a_n(n)$. (page 56)

A4. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i ? (page 57)

A5. Let $I_m = \int_0^{2\pi} \cos(x) \cos(2x) \cdots \cos(mx) dx$. For which integers m , $1 \leq m \leq 10$, is $I_m \neq 0$? (page 58)

A6. If $p(x) = a_0 + a_1x + \cdots + a_mx^m$ is a polynomial with real coefficients a_i , then set

$$\Gamma(p(x)) = a_0^2 + a_1^2 + \cdots + a_m^2.$$

Let $f(x) = 3x^2 + 7x + 2$. Find, with proof, a polynomial $g(x)$ with real coefficients such that

(i) $g(0) = 1$, and

(ii) $\Gamma(f(x)^n) = \Gamma(g(x)^n)$

for every integer $n \geq 1$.

(page 59)

B1. Let k be the smallest positive integer with the following property:

There are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients.

Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved. (page 60)

B2. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx} (f_{n+1}(x)) = (n+1)f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes. (page 61)

B3. Let

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n) . (page 61)

B4. Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference of C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x - and y -axes with diagonal pq . What is the probability that no point of R lies outside of C ? (page 62)

B5. Evaluate $\int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})} dt$. You may assume that $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$. (page 62)

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088385807X - The William Lowell Putnam Mathematical Competition 1985-2000: Problem, Solutions, and Commentary

Kiran S. Kedlaya, Bjorn Poonen and Ravi Vakil

Excerpt

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Problems: The Forty-Sixth Competition (1985)

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B6. Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^r \operatorname{tr}(M_i) = 0$, where $\operatorname{tr}(A)$ denotes the trace of the matrix A . Prove that $\sum_{i=1}^r M_i$ is the $n \times n$ zero matrix. (page 63)

**The Forty-Seventh William Lowell Putnam Mathematical Competition
December 6, 1986**

*Questions Committee: Richard P. Stanley,
Harold M. Stark, and Abraham P. Hillman
See page 36 for hints.*

A1. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$. (page 65)

A2. What is the units (i.e., rightmost) digit of $\left\lfloor \frac{10^{2000}}{10^{100}+3} \right\rfloor$? Here $\lfloor x \rfloor$ is the greatest integer $\leq x$. (page 65)

A3. Evaluate $\sum_{n=0}^{\infty} \operatorname{Arccot}(n^2 + n + 1)$, where $\operatorname{Arccot} t$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot \theta = t$. (page 65)

A4. A *transversal* of an $n \times n$ matrix A consists of n entries of A , no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:

- (a) Each entry $\alpha_{i,j}$ of A is in the set $\{-1, 0, 1\}$.
- (b) The sum of the n entries of a transversal is the same for all transversals of A .

An example of such a matrix A is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,$$

where the a_i 's and b_i 's are rational numbers. (page 67)

A5. Suppose $f_1(x), f_2(x), \dots, f_n(x)$ are functions of n real variables $x = (x_1, \dots, x_n)$ with continuous second-order partial derivatives everywhere on \mathbb{R}^n . Suppose further that there are constants c_{ij} such that

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$$

for all i and j , $1 \leq i \leq n$, $1 \leq j \leq n$. Prove that there is a function $g(x)$ on \mathbb{R}^n such that $f_i + \partial g / \partial x_i$ is linear for all i , $1 \leq i \leq n$. (A linear function is one of the form

$$a_0 + a_1 x_1 + a_2 x_2 + \dots + a_n x_n.)$$

(page 68)

A6. Let a_1, a_2, \dots, a_n be real numbers, and let b_1, b_2, \dots, b_n be distinct positive integers. Suppose there is a polynomial $f(x)$ satisfying the identity

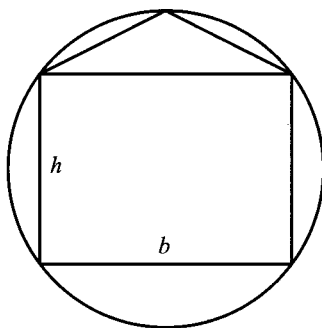
$$(1-x)^n f(x) = 1 + \sum_{i=1}^n a_i x^{b_i}.$$

Problems: The Forty-Seventh Competition (1986)

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Find a simple expression (not involving any sums) for $f(1)$ in terms of b_1, b_2, \dots, b_n and n (but independent of a_1, a_2, \dots, a_n). (page 69)

B1. Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown. For what value of h do the rectangle and triangle have the same area?



(page 70)

B2. Prove that there are only a finite number of possibilities for the ordered triple $T = (x - y, y - z, z - x)$, where x, y , and z are complex numbers satisfying the simultaneous equations

$$x(x - 1) + 2yz = y(y - 1) + 2zx = z(z - 1) + 2xy,$$

and list all such triples T .

(page 71)

B3. Let Γ consist of all polynomials in x with integer coefficients. For f and g in Γ and m a positive integer, let $f \equiv g \pmod{m}$ mean that every coefficient of $f - g$ is an integral multiple of m . Let n and p be positive integers with p prime. Given that f, g, h, r , and s are in Γ with $rf + sg \equiv 1 \pmod{p}$ and $fg \equiv h \pmod{p}$, prove that there exist F and G in Γ with $F \equiv f \pmod{p}$, $G \equiv g \pmod{p}$, and $FG \equiv h \pmod{p^n}$. (page 71)

B4. For a positive real number r , let $G(r)$ be the minimum value of $|r - \sqrt{m^2 + 2n^2}|$ for all integers m and n . Prove or disprove the assertion that $\lim_{r \rightarrow \infty} G(r)$ exists and equals 0. (page 72)

B5. Let $f(x, y, z) = x^2 + y^2 + z^2 + xyz$. Let $p(x, y, z)$, $q(x, y, z)$, $r(x, y, z)$ be polynomials with real coefficients satisfying

$$f(p(x, y, z), q(x, y, z), r(x, y, z)) = f(x, y, z).$$

Prove or disprove the assertion that the sequence p, q, r consists of some permutation of $\pm x, \pm y, \pm z$, where the number of minus signs is 0 or 2. (page 73)

B6. Suppose A, B, C, D are $n \times n$ matrices with entries in a field F , satisfying the conditions that AB^t and CD^t are symmetric and $AD^t - BC^t = I$. Here I is the $n \times n$ identity matrix, and if M is an $n \times n$ matrix, M^t is the transpose of M . Prove that $A^tD - C^tB = I$. (page 74)

**The Forty-Eighth William Lowell Putnam Mathematical Competition
December 5, 1987**

Questions Committee: Harold M. Stark, Abraham P. Hillman, and Gerald A. Heuer
See page 37 for hints.

A1. Curves A , B , C , and D are defined in the plane as follows:[†]

$$A = \left\{ (x, y) : x^2 - y^2 = \frac{x}{x^2 + y^2} \right\},$$

$$B = \left\{ (x, y) : 2xy + \frac{y}{x^2 + y^2} = 3 \right\},$$

$$C = \{ (x, y) : x^3 - 3xy^2 + 3y = 1 \},$$

$$D = \{ (x, y) : 3x^2y - 3x - y^3 = 0 \}.$$

Prove that $A \cap B = C \cap D$. (page 76)

A2. The sequence of digits

123456789101112131415161718192021...

is obtained by writing the positive integers in order. If the 10^n th digit in this sequence occurs in the part of the sequence in which the m -digit numbers are placed, define $f(n)$ to be m . For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$. (page 76)

A3. For all real x , the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real x , must $f'(x) > 0$ for all real x ? Explain.

(b) If $f'(x) > 0$ for all real x , must $f(x) > 0$ for all real x ? Explain. (page 78)

A4. Let P be a polynomial, with real coefficients, in three variables and F be a function of two variables such that

$$P(ux, uy, uz) = u^2 F(y - x, z - x) \quad \text{for all real } x, y, z, u,$$

and such that $P(1, 0, 0) = 4$, $P(0, 1, 0) = 5$, and $P(0, 0, 1) = 6$. Also let A, B, C be complex numbers with $P(A, B, C) = 0$ and $|B - A| = 10$. Find $|C - A|$. (page 78)

A5. Let

$$\vec{G}(x, y) = \left(\frac{-y}{x^2 + 4y^2}, \frac{x}{x^2 + 4y^2}, 0 \right).$$

Prove or disprove that there is a vector-valued function

$$\vec{F}(x, y, z) = (M(x, y, z), N(x, y, z), P(x, y, z))$$

with the following properties:

[†] The equations defining A and B are indeterminate at $(0, 0)$. The point $(0, 0)$ belongs to neither.

Problems: The Forty-Eighth Competition (1987)

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- (i) M, N, P have continuous partial derivatives for all $(x, y, z) \neq (0, 0, 0)$;
 (ii) $\text{Curl } \vec{F} = \vec{0}$ for all $(x, y, z) \neq (0, 0, 0)$;
 (iii) $\vec{F}(x, y, 0) = \vec{G}(x, y)$.

(page 79)

A6. For each positive integer n , let $a(n)$ be the number of zeros in the base 3 representation of n . For which positive real numbers x does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?

(page 79)

B1. Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)} dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

(page 80)

B2. Let r, s , and t be integers with $0 \leq r, 0 \leq s$, and $r + s \leq t$. Prove that

$$\frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \cdots + \frac{\binom{s}{s}}{\binom{t}{r+s}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

(Note: $\binom{n}{k}$ denotes the binomial coefficient $\frac{n(n-1)\cdots(n+1-k)}{k(k-1)\cdots 3\cdot 2\cdot 1}$.) (page 81)

B3. Let F be a field in which $1 + 1 \neq 0$. Show that the set of solutions to the equation $x^2 + y^2 = 1$ with x and y in F is given by $(x, y) = (1, 0)$ and

$$(x, y) = \left(\frac{r^2 - 1}{r^2 + 1}, \frac{2r}{r^2 + 1} \right),$$

where r runs through the elements of F such that $r^2 \neq -1$. (page 83)

B4. Let $(x_1, y_1) = (0.8, 0.6)$ and let $x_{n+1} = x_n \cos y_n - y_n \sin y_n$ and $y_{n+1} = x_n \sin y_n + y_n \cos y_n$ for $n = 1, 2, 3, \dots$. For each of $\lim_{n \rightarrow \infty} x_n$ and $\lim_{n \rightarrow \infty} y_n$, prove that the limit exists and find it or prove that the limit does not exist. (page 85)

B5. Let O_n be the n -dimensional vector $(0, 0, \dots, 0)$. Let M be a $2n \times n$ matrix of complex numbers such that whenever $(z_1, z_2, \dots, z_{2n})M = O_n$, with complex z_i , not all zero, then at least one of the z_i is not real. Prove that for arbitrary real numbers r_1, r_2, \dots, r_{2n} , there are complex numbers w_1, w_2, \dots, w_n such that

$$\text{Re} \left[M \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \right] = \begin{pmatrix} r_1 \\ \vdots \\ r_{2n} \end{pmatrix}.$$

(Note: if C is a matrix of complex numbers, $\text{Re}(C)$ is the matrix whose entries are the real parts of the entries of C .) (page 85)

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Excerpt

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B6. Let F be the field of p^2 elements where p is an odd prime. Suppose S is a set of $(p^2 - 1)/2$ distinct nonzero elements of F with the property that for each $a \neq 0$ in F , exactly one of a and $-a$ is in S . Let N be the number of elements in the intersection $S \cap \{2a : a \in S\}$. Prove that N is even. (page 86)

Problems: The Forty-Ninth Competition (1988)

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**The Forty-Ninth William Lowell Putnam Mathematical Competition
December 3, 1988**

Questions Committee: Abraham P. Hillman, Gerald A. Heuer, and Paul R. Halmos

See page 38 for hints.

A1. Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region R and find its area. (page 88)

A2. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b) . (page 88)

A3. Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \csc \frac{1}{n} - 1 \right)^x$$

converges.

(page 89)

A4.

(a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?

(b) What if “three” is replaced by “nine”?

Justify your answers.

(page 90)

A5. Prove that there exists a *unique* function f from the set \mathbb{R}^+ of positive real numbers to \mathbb{R}^+ such that

$$f(f(x)) = 6x - f(x) \quad \text{and} \quad f(x) > 0 \quad \text{for all } x > 0.$$

(page 92)

A6. If a linear transformation A on an n -dimensional vector space has $n + 1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer. (page 93)

B1. A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y , and z positive integers. (page 94)

B2. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$. (page 95)

B3. For every n in the set $\mathbb{Z}^+ = \{1, 2, \dots\}$ of positive integers, let r_n be the minimum value of $|c - d\sqrt{3}|$ for all nonnegative integers c and d with $c + d = n$. Find, with proof, the smallest positive real number g with $r_n \leq g$ for all $n \in \mathbb{Z}^+$. (page 96)

B4. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$. (page 97)

B5. For positive integers n , let \mathbf{M}_n be the $2n+1$ by $2n+1$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1 . Find, with proof, the rank of \mathbf{M}_n . (According to one definition, the rank of a matrix is the largest k such that there is a $k \times k$ submatrix with nonzero determinant.)

One may note that

$$\mathbf{M}_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_2 = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

(page 97)

B6. Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t the number $at + b$ is a triangular number if and only if t is a triangular number. (The triangular numbers are the $t_n = n(n+1)/2$ with n in $\{0, 1, 2, \dots\}$.) (page 100)