# Contents

**with brief chapter descriptions**

**1 Beauty and the Beast** **1**

Philosopher, attempting to learn about conics, quickly becomes frustrated by the lack of symmetry and many exceptions encountered in the standard treatments. He pours his heart out to Teacher, whose patience and broad knowledge help the trio arrive at the unification and greater depth inherent in the subject. This is usually done by insisting that things be consistent, symmetric and beautiful.

**2 Life at Infinity** **27**

It is discovered that an important source of problems lies in the traditional "viewing screen" of conics, the plane. By drawing identical ellipses on opposite sides on a sphere and then rotating the sphere, one sees that as part of one ellipse disappears over the horizon, another part of its mate appears over the opposite horizon, thus creating the two branches of a hyperbola. The horizon, in mathematical terms, turns out to be the "line at infinity."

**3 How to Gift-Wrap a Hyperbola** **45**

Every ellipse symmetrically rests inside its "standard rectangle," touching the midpoint of each of the rectangle's four sides. If ellipses and hyperbolas are so unified, then why don't we see a similar box surrounding a hyperbola? Standard treatments never answer this. By looking at conics in projective space, it becomes apparent that the familiar diagonals, used in drawing a hyperbola's asymptotes, are in fact the two missing rectangle sides.

**4 The Cube** **57**

The sphere is compressed to form a cube, and the ellipse together with the two standard hyperbola views (corresponding, roughly, to equations $x^2 + y^2 = 1$, $x^2 - y^2 = 1$ and $-x^2 + y^2 = 1$) now appear as circles on the three pairs of opposite faces. This not only highlights inherent symmetry, but the flat faces of the cube make viewing curves drawn on them easy and familiar.

**5 The Other Foci: A Well-Kept Secret** **65**

A ideally-reflecting ellipse has two "sweet points," where light or sound emanating from one focal point converges to the other. Our trio of characters makes its first really major

discovery: by expanding the context from the real domain to the domain of complex numbers, they discover a new pair of focal points lying on the imaginary axis. All the traditional properties of focal-point pairs are shown to hold for this new pair.

**6 Are Hyperbolas *Really* Ellipses? 89**

By this time, the trio has fairly well established that hyperbolas and ellipses are one and the same—just looked at from different perspectives. Our friends now perform a series of reality checks, showing that computations carried out for an ellipse directly translate to "mirror" computations for a hyperbola, and vice-versa.

**7 Stakes and Strings 103**

Tie each end of a piece of string to a thumb tack and push the tacks into a drawing board, leaving some slack in the string. Then use a pen to keep the string taut and swing the pen around. This draws an ellipse, and since the string doesn't stretch, the two string segments always sum to the same constant. To get a hyperbola, you subtract the two string lengths. But if hyperbolas are essentially ellipses, then why must you subtract? The trio discovers that graphing the distance from either focus to an ellipse point produces a line. This leads to signed distances, and not only solves the ellipse-hyperbola conundrum, but leads to an exciting discovery: as you move beyond an ellipse vertex, one segment keeps on giving to the other, and you climb up a hyperbola branch in the Minkowski plane. When one string has donated all its real length to the other, the second string, having zero length, lies on a null line.

**8 Directrices, New and Old 121**

One way the ancient Greeks defined conics was via a fixed point (focus) and line (directrix). Philosopher's aesthetic sense tells him that since there's a directrix associated with each focus, and since they'd found a new pair of foci, there should also exist a second pair of directrices. By carefully constructing drawings of 3-D slices of four-space, they succeed in uncovering the suspected new pair of directrices, and they show that these directrices have properties analogous to traditional ones. This removes some further exceptions that had been bothering Philosopher.

**9 Conics in General Position 143**

Any ellipse can be looked at as a suitably-stretched circle. Is the same true of hyperbolas? The trio uses the power of linear algebra to show that it is. They come to realize that symmetric matrices can have imaginary entries, and find that it's precisely stretching in imaginary directions that produces hyperbolas.

**10 A Beautiful Mathematical Universe 169**

Teacher explains how to add points at infinity to $\mathbb{C}^2$, arriving at "complex projective 2-space," $\mathbb{P}^2(\mathbb{C})$. This turns out to be the best setting in which to consider conics. In a real, finite setting, intersection points can seem to vanish. For example, take two "crossing ellipses," meeting in four points. As you move one ellipse but not the other, those four points decrease to two after a while, and ultimately, to no points of intersection. The trio discovers that the "disappearing points" never actually disappear at all—they simply move into complex space, sometimes even escaping to infinity. The larger setting keeps track of all points, and this leads to simple, consistent results.

**11 A Most Excellent Theorem 183**

In moving one of the above "crossed ellipses" precisely when the number of points decreases, the ellipses become tangent to each other, two points coalescing into one. By regarding such merged points as "multiple"—lying on top of each other—one can prove a very general and beautiful extension of the fundamental theorem of algebra. The trio explores some of the consequences of this generalization, including Pascal's and Brianchon's theorems.

**12 The Big View 215**

At this stage, the trio has extended its vistas to the larger "living space" of $\mathbb{P}^2(\mathbb{C})$. Since there are now more solutions to the equations defining conics, the conics not only include more points, they also now live in four dimensions. Can we peek into the fourth dimension to see what these more complete conics look like? By gluing together 3-D slices of 4-space, the trio finds that these conics are topologically spheres.

**13 Curvature 235**

To gain a better understanding of just how conics are situated in four dimensions, the trio works out curvature formulas for general smooth surfaces.

**14 Curvature of Conics 269**

The formulas obtained in Chapter 13 are now applied to conics themselves. The trio discovers that the usual circle, as it lies on the full conic—a topological sphere in 4-space—does indeed have constant curvature. But that curvature is negative, not positive. Locally, the circle looks like the innermost circle drawn on an inner tube.

**15 Photons and Conics 283**

One problem has bothered Philosopher ever since he first began studying conics: if you release a photon from one focus of a reflecting ellipse, then when it hits the ellipse, it bounces off and heads directly to the other focus; but try that with a hyperbola, and the photon, after reflecting, always *heads the wrong way, toward "outer space"!* Textbooks never address this issue, but it's a problem Philosopher finds he simply must resolve. The trio discovers that it is once again the plane that causes the difficulty. Replace the plane with a very large sphere, and let the photon do the physically correct thing. The photon will then travel along a long geodesic (a "great circle"), finally arriving at the other focus, as it should.

**16 How Conics Solved a 2000-Year-Old Problem 305**

One of the most torturous scientific odysseys was the centuries-long journey to unravel the mystery of planetary motion. Kepler, of course, discovered that their orbits are ellipses. Newton found something that Kepler missed: heavenly bodies may also move along parabolas or hyperbolas—conic sections. His masterpiece *Principia* included a way to find the equation for Kepler's ellipse from a bare minimum of observational data. After reviewing Newton's arguments, the chapter gives the explicit equation of an elliptical orbit in terms of the "launch data."

**17 Waves and Conics** **341**

Take an electric tuning fork having a needle soldered to the end of each prong, and dip the vibrating needle points into a vat of quiet water. The circular ripples created by each vibrating needle interfere, forming a system of hyperbolic "interference fringes" that are broader than the circular background ripples. In the practical world, light from a star passing through two slits in a baffle creates a series of light-and-dark fringes, and the apparatus can be arranged to make these fringes far larger than the wavelength of light. This is the basis of modern radio astronomy, which has provided pictures of unprecedented detail of galaxies near the boundary of the universe, of quasars, and of details near the black hole at the center of our Milky Way. The trio explores these timely and exciting ideas.

**Appendix 1 Some Conics Formulas** **359**

A general conic has five degrees of freedom (three if its location is fixed—for example, if its center is at the origin). Since these parameters uniquely determine the conic, they uniquely determine measurements like eccentricity, principal-axis lengths, latus rectum, area, distance from center to a focus, distance from center to a directrix, and principal curvatures. We give formulas for these measurements for eight different ways of specifying a conic: intersection of a cone and plane; polar equation; focus-directrix-eccentricity; stake-and-string construction; parametric equations of a conic; conic defined by a quadratic form; path of a celestial body; conic through five points.

**Appendix 2 Topology: A Quick Handshake** **379**

Elementary topological notions appear at several places in the text. This appendix gives the necessary topological definitions, with examples, for those who need them.

**Suggestions for Further Reading** **393**

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