Conics

Library of Congress Catalog Card Number 2005923496

ISBN-13 978-0-88385-335-1 hardback
ISBN-10 0-88385-335-3 hardback

Printed in the United States of America

Current printing (last digit):
10 9 8 7 6 5 4 3 2 1

The Dolciani Mathematical Expositions

NUMBER TWENTY-NINE

Conics

Keith Kendig

Published and distributed by
The Mathematical Association of America

DOLCIANI MATHEMATICAL EXPOSITIONS

Cambridge University Press
0883853353 - Conics
Keith Kendig
Frontmatter
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The DOLCIANI MATHEMATICAL EXPOSITIONS series of the Mathematical Association of America was established through a generous gift to the Association from Mary P. Dolciani, Professor of Mathematics at Hunter College of the City University of New York. In making the gift, Professor Dolciani, herself an exceptionally talented and successful expositor of mathematics, had the purpose of furthering the ideal of excellence in mathematical exposition.

The Association, for its part, was delighted to accept the gracious gesture initiating the revolving fund for this series from one who has served the Association with distinction, both as a member of the Committee on Publications and as a member of the Board of Governors. It was with genuine pleasure that the Board chose to name the series in her honor.

The books in the series are selected for their lucid expository style and stimulating mathematical content. Typically, they contain an ample supply of exercises, many with accompanying solutions. They are intended to be sufficiently elementary for the undergraduate and even the mathematically inclined high-school student to understand and enjoy, but also to be interesting and sometimes challenging to the more advanced mathematician.

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In memory of
Hassler Whitney

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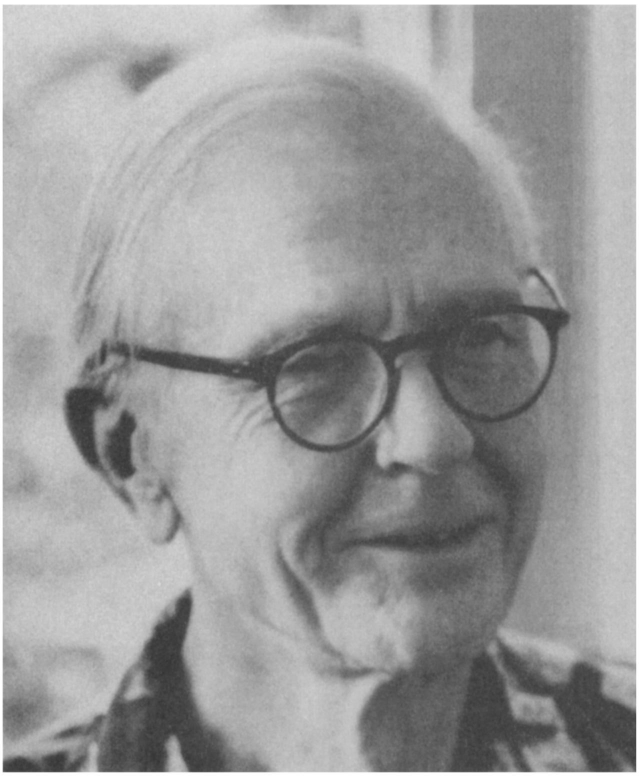

Hassler Whitney
1907–1989

Preface

Beauty is only skin deep, but
Ugliness goes all the way
to the bone.
—Anonymous

In mathematics,
Ugliness is only skin deep, but
Beauty goes all the way
to the bone.
—*Philosopher*, in this book

This book arose from a simple philosophy: in mathematics, the fundamental ought to be beautiful. Conics are certainly basic, arising in many areas of mathematics as well as in physics. Their theory should be consistent, unified and beautiful. Throughout years of teaching conics at various levels—from pre calculus to graduate engineering—I all too often bumped into a lack of consistency, unity and beauty. I found that much of the traditional treatment is more of a collage of smaller designs rather than a unified whole. For example, formulas and routines used for ellipses are different for hyperbolas, and different yet for parabolas.

There also seem to be gaps in the general body of knowledge about conics. As just one example, after we talked in class about tilted conics $Ax^2+Bxy+Cy^2 = 1$, a student asked if there is a formula giving the area of a tilted ellipse. What a *natural*! The coefficients completely determine the tilted conic, so why not a formula in terms of A, B and C? I'd never seen such a formula, so I ended up figuring it out: Area $= 2\pi i/\sqrt{B^2 - 4AC}$. I eventually noticed that if the tilted conic is not an ellipse but a hyperbola, plugging in the coefficients A, B and C still produces an answer. Now, what on earth did "the area of a hyperbola" mean? The formula was supplying information about *something*. What was that something?

When it was suggested that I write a book for the MAA, I saw an opportunity to finally "do conics right." Perhaps I could at last see conics in a more unified way, and fill in some of those gaps. This book is my answer.

The book is written as a "mini-seminar" involving three characters. To an extent their question-and-answer process mirrors my own way of working—lots of examples, all sorts of little debates on why things don't seem beautiful, and what to do about that state of affairs.

Our characters have varying amounts of mathematical background and quite different personalities, but this lends a synergism to their explorations. Here they are:

Philosopher is by far the most iconoclastic of the three, and progress couldn't take place without him. He's the main source of important questions, and these generally come to him because of his unusual radar system. Like a good detective, he picks up subtle clues often leading to larger consequences. Think, for example, of a paleontologist working at a site, whose experienced eye picks up a small outcropping that turns out to be a part of a buried dinosaur.

Philosopher's "outcroppings" are things like arbitrariness, lack of symmetry and exceptions, all frequently lying buried in centuries-old nomenclature, specific-use formulas or points of view born of an age when understanding was more limited and expectations different.

Teacher is mathematically the most broadly educated of the three, but tends toward the traditional. But with his extensive background, he is often in a position to make important connections, and can see a broad landscape developing. He has an open mind and is quick to see value in another's point of view. He is, in fact, often inspired by Philosopher's sometimes outrageous questions. And when an idea requires a fuller explanation or clarification, he is only too happy to oblige.

Student is, as his name implies, the least experienced of the trio. But his fresh spirit and uncluttered mind often prove very useful. He appreciates the power of concrete examples and often asks for them; these can lead to important insights. When the others seem to get just a bit too highfalutin or abstract, it is he who brings them back to *terra firma*.

Our trio attempts to rectify their problems one by one, and their journeys lead them to an increasingly beautiful and expansive vista of conics.

What are the prerequisites? This book is written mainly for upper-division students and beyond, although less experienced students will learn something new. It could form the basis of a senior capstone course. The book assumes a course in calculus, basic linear algebra, some differential equations and at least a popular exposure to "coffee-cup-and-donut topology." For those who don't know what a topological space is, Appendix 2 fills in the basics. Also, there are a few exercises requiring a rudimentary ability to use Maple, Mathematica or other software of this type.

I've attempted to write the book so that if you don't understand something, and if the above appendices don't help, then it's okay just to skip it—that won't stand in the way of getting the main idea.

And those boxes? There are shaded boxes scattered throughout the text. These contain anything from historical tidbits, to comments illuminating or synopsizing what we've been talking about, to the occasional pearl of wisdom. Mostly, those bits of wisdom are intended for everyone, but occasionally one is aimed at the graduate student or beyond.

Acknowledgments. I want to express my appreciation to a number of people who helped me with this book, directly or indirectly. My wife Joan has been a constant and helpful companion during the nearly two years I've spent writing this book. She read the entire manuscript, making many suggestions for improvement, thanks to her English-major and book-writing background. I so much appreciate everything! And my son, Chris, gave me a tremendous amount of help: he put together a web site for me, helped me learn about and make applets, and supplied 24/7 tech support. All that followed our building together a super-fast, capacious and smoothly-working computer which eventually earned the house-

hold nickname "Honcho." Out of sight and sound in the basement, with connections to the first floor, it has quietly made the many technical aspects of writing a book almost fun.

Many, many thanks to Don Albers who approached me about writing a book after the Ford Prize presentation. I was sort of in another world, and he must have sensed it. But he kept at it until I finally woke up and said to him, "You're *serious*, aren't you?" The result is this book. Throughout the entire project, he has been a constant and upbeat "cheerleader"—a real friend.

I also owe a big debt of gratitude to Dan Velleman, the outstanding editor of the Dolciani Series. He read carefully for content and understanding, uncovering intellectual glitches, including not only things I said wrong, but things I didn't say but should have. His comments were to-the-point and insightful. I was fortunate to have his great care go into this book.

I knew there would be many figures in my nascent *opus*, and I wanted to do things right: there *had* to be some "best way" to create all those figures. In writing short papers, I'd muse from time to time about that problem, but had never encountered the right person to ask—that is, until I happened to mention it to Beverly Ruedi, manager of electronic production at MAA. I'll never forget the experience. After that very enlightening phone conversation, Joan and I dashed down to the university bookstore, arriving about ten minutes before closing, and got a copy of Adobe Illustrator 10. I soon began to sense the absolutely incredible power of this program working in tandem with software such as Mathematica, which creates mathematically correct vector-based graphics. Such output seamlessly exports to Illustrator for polishing in almost any direction. Of course, there was a learning curve for me, but as time went on, I felt my "illustrating powers" grow naturally. For me, it's been a good experience, and I hope you like the result.

It is also a pleasure to say "Thank you!" to Elaine Pedreira, who efficiently and skillfully performs the multitude of tasks in shepherding MAA manuscripts through production. She edits copy for ads, contracts for compositors, artists and printers, and solves problems—big and little—that always have a way of arising unbidden. It has been my good fortune to have benefited from her long and broad experience.

My colleagues at Cleveland State have been very supportive. John Oprea fielded what must have seemed like an endless stream of questions from me as I struggled to fill in yawning chasms in my knowledge of differential geometry. He carefully read Chapters 13 and 14 and made many helpful suggestions. Barbara Margolius read other parts of the manuscript and gave me the main impetus to include applets. That meant learning something new, and I want to thank both Sally Shao and Felipe Martins for patiently helping me through some tricky aspects in making applets work.

In connection to applets, "JavaView" is a marvelous graphics add-on to Maple or Mathematica, developed by Steve Durango, Konrad Polthier and others in Germany. Professor Polthier answered my many e-mail inquiries promptly and informatively, and it is a real pleasure to say "Thank you!" for his tremendous support.

Some of my former math teachers and colleagues have left an indelible mark. It was my high school teacher Robert Crawford who inspired me to go on in mathematics. I'll never forget the excitement I felt the time he explained how dragging an ellipse up to the line at infinity makes a parabola, and crossing that line makes a hyperbola. Wow! His enthusiasm and love of mathematics were contagious. He was blunt, outspoken and tremendously supportive. As an original, mold-busting character, it's not entirely outrageous to sum up Bob Crawford this way: Feynman was the Feynman at CalTech; Crawford was the Feynman at Santa Monica High School.

Angus Taylor was my calculus teacher at UCLA; watching his simple, personable approach to solving calculus problems made a lifelong impression. Once after class, when I wasn't too clear on some point, he asked me to explain my understanding of it as best I could. I struggled, he patiently listened, and after a while the problem began to solve itself. Then he asked me if I had a younger brother. No, I didn't. "Well, make one up, and explain things to him, in his terms, so he finally understands!" I never forgot my teacher's advice, and some readers will pick up his influence in this book.

Basil Gordon was my thesis advisor at UCLA. I first met him in a class on tensor methods given by Rolf Nevanlinna, and it was soon clear that this young faculty member was not only uncommonly bright, but had a special talent for clearly explaining things when I got confused. It turned out that the next year he was to teach a course on algebraic geometry, and I jumped at the chance to enroll in it. He turned me on with his very first lecture —drag two crossing ellipses, and they still intersect in four points—it's just that the points go off into complex space of four real dimensions. If we could see in 4-D, we could follow them along. His many subsequent models-of-clarity lectures made my going into algebraic geometry all but inevitable.

Basil has remained a true friend and supporter throughout the many years. He not only proofread *Elementary Algebraic Geometry*, but this book as well. I was so happy to benefit once again from a rare combination of a broad and deep mathematical background, sharp editorial eye, and a wonderful sense of language. His careful reading has resulted in many dozens of corrections and improvements, and his unstinting care has indeed left a strong imprint on *Conics*.

I worked with Hassler Whitney for two years at the Institute for Advanced Study, and the mark he left on me goes deep. I guess I'm attracted to unconventional, original sorts, and Hass was surely that. His undergraduate days were at Yale, so one might expect that here was just about the most incredible math major ever. But his major was music, not math. Well, then, he must have taken all sorts of math courses, and.... Actually, he took almost no math courses. Mathematically, he was largely self-taught.

Anyway, one day, in his office, I happened to mention Bézout's theorem, which basically says that two curves of degree n and m intersect in nm points. He says he never heard of it (Bézout's theorem is in fact highly under-appreciated), and seems galvanized by it. He jumps up and heads toward the blackboard, saying "Let's see if I can disprove that!" *Disprove it?!* "Wait a minute!" I say, "That theorem is nearly two centuries old! You can't disprove anything...really..." As he begins working on some counterexamples at the blackboard, I see that my well-meant words are simply static.

His first tries were easy to demolish, but he was a fast learner, and ideas soon surfaced about the complex line at infinity, and how to count multiple points of intersection. After a while, it got harder for me to justify the theorem, and when he asked, "What about two concentric circles?" I had no answer. He argued his way through, and eventually found all four points. Finally he was satisfied, and the piece of chalk was given a rest. He backed away from the blackboard and said, "Well, well—that *is* quite a theorem, isn't it?"

I think I mostly kept my cool during all this, but after I left his office, I realized I was pretty shaken. I remember thinking to myself, "Golly, Kendig, you just saw how one of the giants does it!" He'd taken the theorem to the mat, wrestled it, and the theorem won. I'd known about that result for at least two years, and I realized that in 15 or 20 minutes, he'd gained a deeper appreciation of it than I'd ever had. In retrospect, it represented a turning point for me: I began to think examples, examples. Whitney worked by finding an example that contained the essential crux of a problem, and then worked relentlessly on it until he cracked it. He left it to others to generalize. It is to Hass that I affectionately dedicate this book.

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with brief chapter descriptions