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Conics

Keith Kendig



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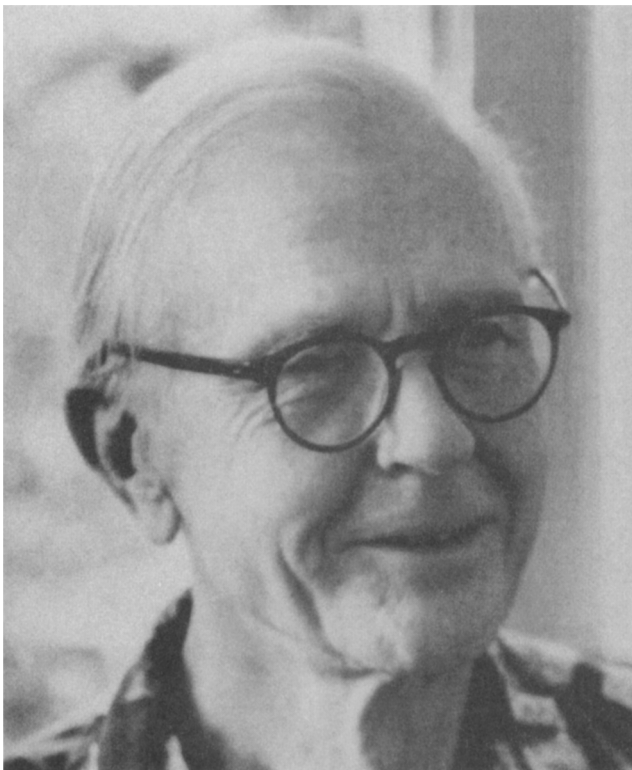
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In memory of
Hassler Whitney

Photo: Herman Landshoff. Courtesy of the Archives of the Institute for Advanced Study.



Hassler Whitney
1907–1989

Preface

*Beauty is only skin deep, but
 Ugliness goes all the way
 to the bone.*

—Anonymous

*In mathematics,
 Ugliness is only skin deep, but
 Beauty goes all the way
 to the bone.*

—*Philosopher, in this book*

This book arose from a simple philosophy: in mathematics, the fundamental ought to be beautiful. Conics are certainly basic, arising in many areas of mathematics as well as in physics. Their theory should be consistent, unified and beautiful. Throughout years of teaching conics at various levels—from pre calculus to graduate engineering—I all too often bumped into a lack of consistency, unity and beauty. I found that much of the traditional treatment is more of a collage of smaller designs rather than a unified whole. For example, formulas and routines used for ellipses are different for hyperbolas, and different yet for parabolas.

There also seem to be gaps in the general body of knowledge about conics. As just one example, after we talked in class about tilted conics $Ax^2 + Bxy + Cy^2 = 1$, a student asked if there is a formula giving the area of a tilted ellipse. What a *natural*! The coefficients completely determine the tilted conic, so why not a formula in terms of A , B and C ? I'd never seen such a formula, so I ended up figuring it out: $\text{Area} = 2\pi i / \sqrt{B^2 - 4AC}$. I eventually noticed that if the tilted conic is not an ellipse but a hyperbola, plugging in the coefficients A , B and C still produces an answer. Now, what on earth did “the area of a hyperbola” mean? The formula was supplying information about *something*. What was that something?

When it was suggested that I write a book for the MAA, I saw an opportunity to finally “do conics right.” Perhaps I could at last see conics in a more unified way, and fill in some of those gaps. This book is my answer.

The book is written as a “mini-seminar” involving three characters. To an extent their question-and-answer process mirrors my own way of working—lots of examples, all sorts of little debates on why things don't seem beautiful, and what to do about that state of affairs.

Our characters have varying amounts of mathematical background and quite different personalities, but this lends a synergism to their explorations. Here they are:

Philosopher is by far the most iconoclastic of the three, and progress couldn't take place without him. He's the main source of important questions, and these generally come to him because of his unusual radar system. Like a good detective, he picks up subtle clues often leading to larger consequences. Think, for example, of a paleontologist working at a site, whose experienced eye picks up a small outcropping that turns out to be a part of a buried dinosaur.

Philosopher's "outcroppings" are things like arbitrariness, lack of symmetry and exceptions, all frequently lying buried in centuries-old nomenclature, specific-use formulas or points of view born of an age when understanding was more limited and expectations different.

Teacher is mathematically the most broadly educated of the three, but tends toward the traditional. But with his extensive background, he is often in a position to make important connections, and can see a broad landscape developing. He has an open mind and is quick to see value in another's point of view. He is, in fact, often inspired by **Philosopher's** sometimes outrageous questions. And when an idea requires a fuller explanation or clarification, he is only too happy to oblige.

Student is, as his name implies, the least experienced of the trio. But his fresh spirit and uncluttered mind often prove very useful. He appreciates the power of concrete examples and often asks for them; these can lead to important insights. When the others seem to get just a bit too highfalutin or abstract, it is he who brings them back to *terra firma*.

Our trio attempts to rectify their problems one by one, and their journeys lead them to an increasingly beautiful and expansive vista of conics.

What are the prerequisites? This book is written mainly for upper-division students and beyond, although less experienced students will learn something new. It could form the basis of a senior capstone course. The book assumes a course in calculus, basic linear algebra, some differential equations and at least a popular exposure to "coffee-cup-and-donut topology." For those who don't know what a topological space is, Appendix 2 fills in the basics. Also, there are a few exercises requiring a rudimentary ability to use Maple, Mathematica or other software of this type.

I've attempted to write the book so that if you don't understand something, and if the above appendices don't help, then it's okay just to skip it—that won't stand in the way of getting the main idea.

And those boxes? There are shaded boxes scattered throughout the text. These contain anything from historical tidbits, to comments illuminating or synopsizing what we've been talking about, to the occasional pearl of wisdom. Mostly, those bits of wisdom are intended for everyone, but occasionally one is aimed at the graduate student or beyond.

Acknowledgments. I want to express my appreciation to a number of people who helped me with this book, directly or indirectly. My wife Joan has been a constant and helpful companion during the nearly two years I've spent writing this book. She read the entire manuscript, making many suggestions for improvement, thanks to her English-major and book-writing background. I so much appreciate everything! And my son, Chris, gave me a tremendous amount of help: he put together a web site for me, helped me learn about and make applets, and supplied 24/7 tech support. All that followed our building together a super-fast, capacious and smoothly-working computer which eventually earned the house-

hold nickname “Honcho.” Out of sight and sound in the basement, with connections to the first floor, it has quietly made the many technical aspects of writing a book almost fun.

Many, many thanks to Don Albers who approached me about writing a book after the Ford Prize presentation. I was sort of in another world, and he must have sensed it. But he kept at it until I finally woke up and said to him, “You’re *serious*, aren’t you?” The result is this book. Throughout the entire project, he has been a constant and upbeat “cheerleader”—a real friend.

I also owe a big debt of gratitude to Dan Velleman, the outstanding editor of the Dolciani Series. He read carefully for content and understanding, uncovering intellectual glitches, including not only things I said wrong, but things I didn’t say but should have. His comments were to-the-point and insightful. I was fortunate to have his great care go into this book.

I knew there would be many figures in my nascent *opus*, and I wanted to do things right: there *had* to be some “best way” to create all those figures. In writing short papers, I’d muse from time to time about that problem, but had never encountered the right person to ask—that is, until I happened to mention it to Beverly Ruedi, manager of electronic production at MAA. I’ll never forget the experience. After that very enlightening phone conversation, Joan and I dashed down to the university bookstore, arriving about ten minutes before closing, and got a copy of Adobe Illustrator 10. I soon began to sense the absolutely incredible power of this program working in tandem with software such as Mathematica, which creates mathematically correct vector-based graphics. Such output seamlessly exports to Illustrator for polishing in almost any direction. Of course, there was a learning curve for me, but as time went on, I felt my “illustrating powers” grow naturally. For me, it’s been a good experience, and I hope you like the result.

It is also a pleasure to say “Thank you!” to Elaine Pedreira, who efficiently and skillfully performs the multitude of tasks in shepherding MAA manuscripts through production. She edits copy for ads, contracts for compositors, artists and printers, and solves problems—big and little—that always have a way of arising unbidden. It has been my good fortune to have benefited from her long and broad experience.

My colleagues at Cleveland State have been very supportive. John Oprea fielded what must have seemed like an endless stream of questions from me as I struggled to fill in yawning chasms in my knowledge of differential geometry. He carefully read Chapters 13 and 14 and made many helpful suggestions. Barbara Margolius read other parts of the manuscript and gave me the main impetus to include applets. That meant learning something new, and I want to thank both Sally Shao and Felipe Martins for patiently helping me through some tricky aspects in making applets work.

In connection to applets, “JavaView” is a marvelous graphics add-on to Maple or Mathematica, developed by Steve Durango, Konrad Polthier and others in Germany. Professor Polthier answered my many e-mail inquiries promptly and informatively, and it is a real pleasure to say “Thank you!” for his tremendous support.

Some of my former math teachers and colleagues have left an indelible mark. It was my high school teacher Robert Crawford who inspired me to go on in mathematics. I’ll never forget the excitement I felt the time he explained how dragging an ellipse up to the line at infinity makes a parabola, and crossing that line makes a hyperbola. Wow! His enthusiasm and love of mathematics were contagious. He was blunt, outspoken and tremendously supportive. As an original, mold-busting character, it’s not entirely outrageous to sum up Bob Crawford this way: Feynman was the Feynman at CalTech; Crawford was the Feynman at Santa Monica High School.

Angus Taylor was my calculus teacher at UCLA; watching his simple, personable approach to solving calculus problems made a lifelong impression. Once after class, when I wasn't too clear on some point, he asked me to explain my understanding of it as best I could. I struggled, he patiently listened, and after a while the problem began to solve itself. Then he asked me if I had a younger brother. No, I didn't. "Well, make one up, and explain things to him, in his terms, so he finally understands!" I never forgot my teacher's advice, and some readers will pick up his influence in this book.

Basil Gordon was my thesis advisor at UCLA. I first met him in a class on tensor methods given by Rolf Nevanlinna, and it was soon clear that this young faculty member was not only uncommonly bright, but had a special talent for clearly explaining things when I got confused. It turned out that the next year he was to teach a course on algebraic geometry, and I jumped at the chance to enroll in it. He turned me on with his very first lecture—drag two crossing ellipses, and they still intersect in four points—it's just that the points go off into complex space of four real dimensions. If we could see in 4-D, we could follow them along. His many subsequent models-of-clarity lectures made my going into algebraic geometry all but inevitable.

Basil has remained a true friend and supporter throughout the many years. He not only proofread *Elementary Algebraic Geometry*, but this book as well. I was so happy to benefit once again from a rare combination of a broad and deep mathematical background, sharp editorial eye, and a wonderful sense of language. His careful reading has resulted in many dozens of corrections and improvements, and his unstinting care has indeed left a strong imprint on *Conics*.

I worked with Hassler Whitney for two years at the Institute for Advanced Study, and the mark he left on me goes deep. I guess I'm attracted to unconventional, original sorts, and Hass was surely that. His undergraduate days were at Yale, so one might expect that here was just about the most incredible math major ever. But his major was music, not math. Well, then, he must have taken all sorts of math courses, and... Actually, he took almost no math courses. Mathematically, he was largely self-taught.

Anyway, one day, in his office, I happened to mention Bézout's theorem, which basically says that two curves of degree n and m intersect in nm points. He says he never heard of it (Bézout's theorem is in fact highly under-appreciated), and seems galvanized by it. He jumps up and heads toward the blackboard, saying "Let's see if I can disprove that!" *Disprove it?!* "Wait a minute!" I say, "That theorem is nearly two centuries old! You can't disprove anything... really..." As he begins working on some counterexamples at the blackboard, I see that my well-meant words are simply static.

His first tries were easy to demolish, but he was a fast learner, and ideas soon surfaced about the complex line at infinity, and how to count multiple points of intersection. After a while, it got harder for me to justify the theorem, and when he asked, "What about two concentric circles?" I had no answer. He argued his way through, and eventually found all four points. Finally he was satisfied, and the piece of chalk was given a rest. He backed away from the blackboard and said, "Well, well—that *is* quite a theorem, isn't it?"

I think I mostly kept my cool during all this, but after I left his office, I realized I was pretty shaken. I remember thinking to myself, "Golly, Kendig, you just saw how one of the giants does it!" He'd taken the theorem to the mat, wrestled it, and the theorem won. I'd known about that result for at least two years, and I realized that in 15 or 20 minutes, he'd gained a deeper appreciation of it than I'd ever had. In retrospect, it represented a turning point for me: I began to think examples, examples. Whitney worked by finding an example that contained the essential crux of a problem, and then worked relentlessly on it until he cracked it. He left it to others to generalize. It is to Hass that I affectionately dedicate this book.

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with brief chapter descriptions

1 Beauty and the Beast 1

Philosopher, attempting to learn about conics, quickly becomes frustrated by the lack of symmetry and many exceptions encountered in the standard treatments. He pours his heart out to Teacher, whose patience and broad knowledge help the trio arrive at the unification and greater depth inherent in the subject. This is usually done by insisting that things be consistent, symmetric and beautiful.

2 Life at Infinity 27

It is discovered that an important source of problems lies in the traditional “viewing screen” of conics, the plane. By drawing identical ellipses on opposite sides on a sphere and then rotating the sphere, one sees that as part of one ellipse disappears over the horizon, another part of its mate appears over the opposite horizon, thus creating the two branches of a hyperbola. The horizon, in mathematical terms, turns out to be the “line at infinity.”

3 How to Gift-Wrap a Hyperbola 45

Every ellipse symmetrically rests inside its “standard rectangle,” touching the midpoint of each of the rectangle’s four sides. If ellipses and hyperbolas are so unified, then why don’t we see a similar box surrounding a hyperbola? Standard treatments never answer this. By looking at conics in projective space, it becomes apparent that the familiar diagonals, used in drawing a hyperbola’s asymptotes, are in fact the two missing rectangle sides.

4 The Cube 57

The sphere is compressed to form a cube, and the ellipse together with the two standard hyperbola views (corresponding, roughly, to equations $x^2 + y^2 = 1$, $x^2 - y^2 = 1$ and $-x^2 + y^2 = 1$) now appear as circles on the three pairs of opposite faces. This not only highlights inherent symmetry, but the flat faces of the cube make viewing curves drawn on them easy and familiar.

5 The Other Foci: A Well-Kept Secret 65

A ideally-reflecting ellipse has two “sweet points,” where light or sound emanating from one focal point converges to the other. Our trio of characters makes its first really major

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discovery: by expanding the context from the real domain to the domain of complex numbers, they discover a new pair of focal points lying on the imaginary axis. All the traditional properties of focal-point pairs are shown to hold for this new pair.

6 Are Hyperbolas Really Ellipses? 89

By this time, the trio has fairly well established that hyperbolas and ellipses are one and the same—just looked at from different perspectives. Our friends now perform a series of reality checks, showing that computations carried out for an ellipse directly translate to “mirror” computations for a hyperbola, and vice-versa.

7 Stakes and Strings 103

Tie each end of a piece of string to a thumb tack and push the tacks into a drawing board, leaving some slack in the string. Then use a pen to keep the string taut and swing the pen around. This draws an ellipse, and since the string doesn’t stretch, the two string segments always sum to the same constant. To get a hyperbola, you subtract the two string lengths. But if hyperbolas are essentially ellipses, then why must you subtract? The trio discovers that graphing the distance from either focus to an ellipse point produces a line. This leads to signed distances, and not only solves the ellipse-hyperbola conundrum, but leads to an exciting discovery: as you move beyond an ellipse vertex, one segment keeps on giving to the other, and you climb up a hyperbola branch in the Minkowski plane. When one string has donated all its real length to the other, the second string, having zero length, lies on a null line.

8 Directrices, New and Old 121

One way the ancient Greeks defined conics was via a fixed point (focus) and line (directrix). Philosopher’s aesthetic sense tells him that since there’s a directrix associated with each focus, and since they’d found a new pair of foci, there should also exist a second pair of directrices. By carefully constructing drawings of 3-D slices of four-space, they succeed in uncovering the suspected new pair of directrices, and they show that these directrices have properties analogous to traditional ones. This removes some further exceptions that had been bothering Philosopher.

9 Conics in General Position 143

Any ellipse can be looked at as a suitably-stretched circle. Is the same true of hyperbolas? The trio uses the power of linear algebra to show that it is. They come to realize that symmetric matrices can have imaginary entries, and find that it’s precisely stretching in imaginary directions that produces hyperbolas.

10 A Beautiful Mathematical Universe 169

Teacher explains how to add points at infinity to \mathbb{C}^2 , arriving at “complex projective 2-space,” $\mathbb{P}^2(\mathbb{C})$. This turns out to be the best setting in which to consider conics. In a real, finite setting, intersection points can seem to vanish. For example, take two “crossing ellipses,” meeting in four points. As you move one ellipse but not the other, those four points decrease to two after a while, and ultimately, to no points of intersection. The trio discovers that the “disappearing points” never actually disappear at all—they simply move

into complex space, sometimes even escaping to infinity. The larger setting keeps track of all points, and this leads to simple, consistent results.

11 A Most Excellent Theorem 183

In moving one of the above “crossed ellipses” precisely when the number of points decreases, the ellipses become tangent to each other, two points coalescing into one. By regarding such merged points as “multiple”—lying on top of each other—one can prove a very general and beautiful extension of the fundamental theorem of algebra. The trio explores some of the consequences of this generalization, including Pascal’s and Brianchon’s theorems.

12 The Big View 215

At this stage, the trio has extended its vistas to the larger “living space” of $\mathbb{P}^2(\mathbb{C})$. Since there are now more solutions to the equations defining conics, the conics not only include more points, they also now live in four dimensions. Can we peek into the fourth dimension to see what these more complete conics look like? By gluing together 3-D slices of 4-space, the trio finds that these conics are topologically spheres.

13 Curvature 235

To gain a better understanding of just how conics are situated in four dimensions, the trio works out curvature formulas for general smooth surfaces.

14 Curvature of Conics 269

The formulas obtained in Chapter 13 are now applied to conics themselves. The trio discovers that the usual circle, as it lies on the full conic—a topological sphere in 4-space—does indeed have constant curvature. But that curvature is negative, not positive. Locally, the circle looks like the innermost circle drawn on an inner tube.

15 Photons and Conics 283

One problem has bothered Philosopher ever since he first began studying conics: if you release a photon from one focus of a reflecting ellipse, then when it hits the ellipse, it bounces off and heads directly to the other focus; but try that with a hyperbola, and the photon, after reflecting, always *heads the wrong way, toward “outer space”*! Textbooks never address this issue, but it’s a problem Philosopher finds he simply must resolve. The trio discovers that it is once again the plane that causes the difficulty. Replace the plane with a very large sphere, and let the photon do the physically correct thing. The photon will then travel along a long geodesic (a “great circle”), finally arriving at the other focus, as it should.

16 How Conics Solved a 2000-Year-Old Problem 305

One of the most torturous scientific odysseys was the centuries-long journey to unravel the mystery of planetary motion. Kepler, of course, discovered that their orbits are ellipses. Newton found something that Kepler missed: heavenly bodies may also move along parabolas or hyperbolas—conic sections. His masterpiece *Principia* included a way to find the equation for Kepler’s ellipse from a bare minimum of observational data. After review-

ing Newton’s arguments, the chapter gives the explicit equation of an elliptical orbit in terms of the “launch data.”

17 Waves and Conics 341

Take an electric tuning fork having a needle soldered to the end of each prong, and dip the vibrating needle points into a vat of quiet water. The circular ripples created by each vibrating needle interfere, forming a system of hyperbolic “interference fringes” that are broader than the circular background ripples. In the practical world, light from a star passing through two slits in a baffle creates a series of light-and-dark fringes, and the apparatus can be arranged to make these fringes far larger than the wavelength of light. This is the basis of modern radio astronomy, which has provided pictures of unprecedented detail of galaxies near the boundary of the universe, of quasars, and of details near the black hole at the center of our Milky Way. The trio explores these timely and exciting ideas.

Appendix 1 Some Conics Formulas 359

A general conic has five degrees of freedom (three if its location is fixed—for example, if its center is at the origin). Since these parameters uniquely determine the conic, they uniquely determine measurements like eccentricity, principal-axis lengths, latus rectum, area, distance from center to a focus, distance from center to a directrix, and principal curvatures. We give formulas for these measurements for eight different ways of specifying a conic: intersection of a cone and plane; polar equation; focus-directrix-eccentricity; stake-and-string construction; parametric equations of a conic; conic defined by a quadratic form; path of a celestial body; conic through five points.

Appendix 2 Topology: A Quick Handshake 379

Elementary topological notions appear at several places in the text. This appendix gives the necessary topological definitions, with examples, for those who need them.

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