Complex Analysis: The Geometric Viewpoint

Second Edition

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To my parents

Acknowledgments

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—S.G.K.

Preface to the Second Edition

The warm reception with which the first edition of this book has been received has been a source of both pride and pleasure. It is a special privilege to have created a "for the record" version of Ahlfors's seminal ideas in the subject. And the geometric viewpoint continues to develop.

In the intervening decade, this author has learned a great deal more about geometric analysis, and his view of the subject has developed and broadened. It seems appropriate, therefore, to bring some new life to these pages, and to set forth a fresh enunciation of the role of curvature in basic complex function theory.

In this new edition, we explain how, in a natural and elementary manner, the hyperbolic disc is a model for the non-Euclidean geometry of Bolyai and Lobachevsky. Later on, we explain the Bergman kernel and provide an introduction to the Bergman metric.

I have many friends and colleagues to thank for their incisive remarks and suggestions about the first edition of this book. I hope that I do them justice in my efforts to implement a second edition. As always, the Mathematical Association of America has been an exemplary publisher and has provided all possible support in the publication process. I offer my humble thanks.

Preface to the First Edition

The modern geometric point of view in complex function theory began with Ahlfors's classic paper [AHL]. In that work it was demonstrated that the Schwarz lemma can be viewed as an inequality of certain differential geometric quantities on the disc (we will later learn that they are curvatures). This point of view—that substantive analytic facts can be interpreted in the language of Riemannian geometry—has developed considerably in the last fifty years. It provides new proofs of many classical results in complex analysis, and has led to new insights as well.

In this monograph we intend to introduce the reader with a standard one semester background in complex analysis to the geometric method. All geometric ideas will be developed from first principles, and only to the extent needed here. No background in geometry is assumed or required.

Chapter 0 gives a bird's eye view of classical function theory of one complex variable. We pay special attention to topics which are developed later in the book from a more advanced point of view. In this chapter we also sketch proofs of the main results, with the hope that the reader can thereby get a feeling for classical methodology before embarking on a study of the geometric method.

Chapter 1 begins a systematic treatment of the techniques of Riemannian geometry, specially tailored to the setting of one complex variable. In order that the principal ideas may be brought out most clearly, we shall concentrate on only a few themes: the Schwarz lemma, the Riemann mapping theorem, normal families, and Picard's theorems. For many readers this will be a first contact with the latter two results. The geometric method provides a particularly cogent explanation of these theorems, and can be contrasted with the more classical proofs which are discussed in Chapter 0. We shall also touch on Fatou-Julia theory, a topic which is rather technical from the analytic standpoint but completely natural from the point of view of geometry.

In Chapter 3 we introduce the Carathéodory and Kobayashi metrics, a device which is virtually unknown in the world of one complex variable. This decision allows us to introduce invariant metrics on arbitrary planar domains without resort to the uniformization theorem. We are then able to give a "differential geometric" interpretation of the Riemann mapping theorem.

The last chapter gives a brief glimpse of several complex variables. Some of the themes which were developed earlier in the book are carried over to two dimensions. Biholomorphic mappings are discussed, and the inequivalence of the ball and bidisc is proved using a geometric argument.

The language of differential geometry is not generally encountered in a first course in complex analysis. It is hoped that this volume will be used as a supplement to such a course, and that it may lead to greater familiarity with the fruitful methodology of geometry.

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