CHAPTER 0

Principal Ideas of Classical Function Theory

1. A Glimpse of Complex Analysis

The purpose of this book is to explain how various aspects of complex analysis can be understood both naturally and elegantly from the point of view of metric geometry. Thus, in order to set the stage for our work, we begin with a review of some of the principal ideas in complex analysis. A good companion volume for this introductory material is [GRK]. See also [BOAS] and [KR3].

Central to the subject are the Cauchy integral theorem and the Cauchy integral formula. From these follow the Cauchy estimates, Liouville's theorem, the maximum principle, Schwarz's lemma, the argument principle, Montel's theorem, and most of the other powerful and elegant results which are basic to the subject. We will discuss these results in essay form. The proofs which we provide are more conceptual than rigorous: *the aim is to depict a flow of ideas rather than absolute mathematical precision.*

We let $z \in \mathbb{C}$ denote a complex variable. If $P \in \mathbb{C}$ and $r > 0$, then we use the standard notation

$$D(P,r) = \{z \in \mathbb{C} : |z - P| < r\},$$
$$\overline{D}(P,r) = \{z \in \mathbb{C} : |z - P| \le r\},$$
$$\partial D(P,r) = \{z \in \mathbb{C} : |z - P| = r\}.$$

We often use the lone symbol D to denote the unit disc $D(0,1)$. A connected open set $U \subseteq \mathbb{C}$ is called a *domain*.

Complex analysis consists of the study of holomorphic functions. Let F be a complex-valued continuously differentiable function (in the sense of multivariable calculus) on a domain U in the complex plane. We write $F = u + iv$ to distinguish the real and imaginary parts of F. Then F is said to be *holomorphic*, or *analytic*, if it satisfies the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

This definition is equivalent to other familiar definitions, such as that in terms of the complex derivative, which we now discuss. If F is a function on a domain U in the complex plane and if $P \in U$, then F is said to possess a *complex derivative*, or to be complex differentiable, at P if

$$F'(P) = \frac{\partial F}{\partial z}(P) = \lim_{z \to P} \frac{F(z) - F(P)}{z - P}$$

exists. The function F is holomorphic on U if F possesses the complex derivative at each point of U.

Another useful approach to complex analytic (or holomorphic) functions is by way of power series: a function on a domain U is holomorphic if it has a convergent power series expansion $\sum_j a_j(z - P)^j$ about each point P of U.

The complex derivative definition of "holomorphic" is of great historical interest. Much effort was expended in the early days of the subject in proving that a function which is complex differentiable at each point of a domain U is in fact automatically continuously differentiable (in the usual sense of multivariable calculus), and from that point it is routine to check that the function satisfies the Cauchy–Riemann equations. The converse implication is a straightforward exercise. So, in the end, either definition is correct. From our perspective the Cauchy–Riemann equations provide the most useful point of view. This assertion will become more transparent as we develop the notion of complex integration.

Definition 1. A C^1, or *continuously differentiable,* curve in a domain $U \subseteq \mathbb{C}$ is a function $\gamma : [a, b] \to U$ from an interval in the real line into U such that γ' exists at each point of $[a, b]$ (in the one-sided sense at the endpoints) and is continuous on $[a, b]$. When there is no danger of confusion, we sometimes use the symbol γ to denote the set of points $\{\gamma(t) : t \in [a, b]\}$ as well as the function from $[a, b]$ to U.

A *piecewise continuously differentiable curve* is a single continuous curve which can be written as a finite union of continuously differentiable curves—Figure 1. A curve is called *closed* if $\gamma(a) = \gamma(b)$. It is called *simple closed* if it is closed and not self-intersecting: $\gamma(s) = \gamma(t)$ and $s \neq t$ together imply either that $s = a$ and $t = b$ or that $s = b$ and $t = a$.

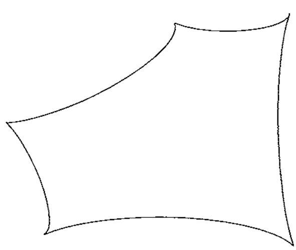

Figure 1.

A simple closed curve γ is said to be *positively oriented* if the region interior to the curve is to the left of the curve while it is being traversed from $t = a$ to $t = b$. See Figure 2. Otherwise it is called *negatively oriented*. If F is a continuous function on our open set U, then

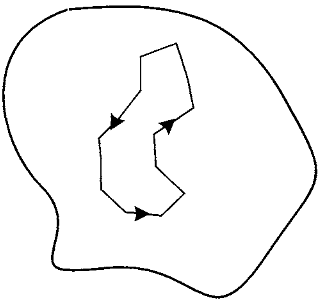

Figure 2.

we define its *complex line integral* over a continuously differentiable curve γ in U to be the quantity

$$\oint_\gamma F(z)\,dz = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt.$$

Here the dot $\cdot$ denotes multiplication of complex numbers.

Notice that, in analogy with the study of directed curves in Stokes's formula, the derivative of the curve is incorporated into the integral. In case γ is only piecewise continuously differentiable, we define

$$\oint_\gamma F(z)\,dz$$

by integrating along each of the continuously differentiable pieces and adding.

Now we may formulate the Cauchy integral theorem. A rigorous treatment of this result requires a discussion of deformation of curves. However, since this is only a review, we may be a bit imprecise. Let γ be a closed curve in a domain U and suppose that γ (more precisely, the *image* of γ) can be continuously deformed to a point within U. We shall call such a curve "topologically trivial (with respect to U)." In Figure 3, γ_2 is topologically trivial but γ_1 is not. We have:

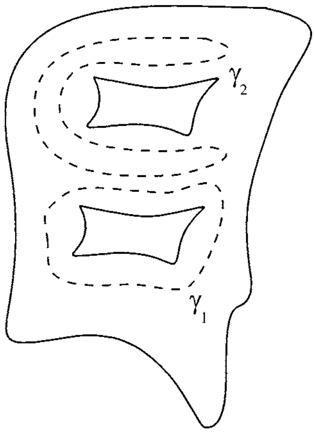

Figure 3.

Theorem 2. *Let F be a holomorphic function on a domain U and let γ be a topologically trivial, piecewise continuously differentiable, closed curve in U. Then*

$$\oint_\gamma F(z)\,dz = 0.$$

This theorem may be proved, using the Cauchy–Riemann equations, as a direct application of Stokes's theorem (see [GRK]). It will tell us, in effect, that a holomorphic function is strongly influenced on an open set by its behavior on the boundary of that set.

Now fix a point P in U and let γ be a positively oriented, topologically trivial, *simple closed curve* in U with P in its interior. Let F be holomorphic on U. By suitable limiting arguments, we may apply the Cauchy integral theorem to the function

$$G(\zeta) \equiv \begin{cases} \dfrac{F(\zeta) - F(P)}{\zeta - P} & \text{if } \zeta \neq P, \\ F'(P) & \text{if } \zeta = P. \end{cases}$$

After some calculations, the result is that

$$F(P) = \frac{1}{2\pi i} \oint_{\gamma} \frac{F(\zeta)}{\zeta - P} \, d\zeta.$$

This is the *Cauchy integral formula*. It shows that a holomorphic function is completely determined in the interior of γ by its behavior on the boundary curve γ itself. From this there quickly flows a wealth of information.

Theorem 3. *Let F be holomorphic on a domain U and let $P \in U$. Assume that the closed disc*

$$\overline{D}(P, r) \equiv \{z : |z - P| \leq r\}$$

is contained in U. Then F may be written on $\overline{D}(P, r)$ as a convergent power series:

$$F(z) = \sum_{j=0}^{\infty} a_j (z - P)^j.$$

The convergence is absolute and uniform on $D(P, r)$.

Thus we see that, in a natural sense, holomorphic functions are generalizations of complex polynomials. The power series expansion is, in general, only local. But for many purposes this is sufficient.

Proof of Theorem 3. Observe that, for $|z - P| < r$ and $|\zeta - P| = r$, we may write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - P} \cdot \frac{1}{1 - \frac{z-P}{\zeta-P}}.$$

Since $|z - P| < r = |\zeta - P|$, we have that

$$\left| \frac{z - P}{\zeta - P} \right| < 1.$$

Thus

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - P} \cdot \sum_{j=0}^{\infty} \left(\frac{z - P}{\zeta - P} \right)^j .$$

Substituting this power series expansion for the Cauchy kernel into the Cauchy integral formula on $\overline{D}(P, r)$ gives the desired power series expansion for the holomorphic function F.

■

As an added bonus, the proof gives us a formula for the series coefficients a_j:

$$a_j = \frac{1}{2\pi i} \oint_\gamma \frac{F(\zeta)}{(\zeta - P)^{j+1}} \, d\zeta .$$

Just as in the theory of Taylor series, it turns out that the coefficients a_j must also be given by

$$a_j = \frac{1}{j!} \left(\frac{\partial^j F}{\partial z^j} \right) (P).$$

We conclude that

$$\left(\frac{\partial^j F}{\partial z^j} \right) (P) = \frac{j!}{2\pi i} \oint_\gamma \frac{F(\zeta)}{(\zeta - P)^{j+1}} \, d\zeta . \qquad (*)$$

Corollary 3.1. (Riemann removable singularities theorem). *We let $\widetilde{F}$ be a holomorphic function on a punctured disc $D'(P, r) \equiv D(P, r) \setminus \{P\}$. If $\widetilde{F}$ is bounded, then $\widetilde{F}$ continues analytically to the entire disc $D(P, r)$. That is, there is a holomorphic function F on $D(P, r)$ such that $F\big|_{D'(P,r)} = \widetilde{F}$.*

Sketch of Proof. Assume without loss of generality that $P = 0$. Consider the function $G(z)$ that is defined to equal $z^2 \cdot \widetilde{F}$ on $D'(P, r)$ and to equal 0 at $P = 0$. Then G is continuously differentiable on $D(P, r)$ and satisfies the Cauchy–Riemann equations.

The leading term of the power series expansion of G about 0 is of the form $a_2 z^2$. Thus the holomorphic function G may be divided by z^2 to define a holomorphic function F on $D(P, r)$ which agrees with $\widetilde{F}$ on $D'(P, r)$. ■

It is a standard fact from the theory of power series that the zeros of a function given by a power series expansion cannot accumulate in the interior of the domain of that function. Thus we have:

Theorem 4. *If F is holomorphic on a domain U, then $\{z \in U : F(z) = 0\}$ has no accumulation point in U.*

This theorem once again bears out the dictum that holomorphic functions are much like polynomials: The zero set of a polynomial $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is discrete, indeed it is finite.

The Cauchy estimates on the derivatives of a holomorphic function in terms of the supremum of the function follow from direct estimation of the formula $(*)$:

Theorem 5. *Let F be a holomorphic function on a domain U that contains the closed disc $\overline{D}(P, R)$. Let M be the supremum of $|F|$ on $\overline{D}(P, R)$. Then the derivatives of F satisfy the estimates*

$$\left| \left(\frac{\partial^j}{\partial z^j} \right) F(P) \right| \leq \frac{j! \cdot M}{R^j}.$$

An immediate corollary of the Cauchy estimates is the fact that if a sequence of holomorphic functions converges then so does the sequence of its derivatives:

Corollary 5.1. *Let $\{F_j\}$ be a sequence of holomorphic functions on a domain Ω. Suppose that the sequence converges uniformly on compact subsets of Ω. Then the sequence $\{F_j'\}$ also converges uniformly on compact subsets of Ω.*

Notice that the Cauchy estimates tell us that if F is bounded on a large disc, then its derivatives are relatively small at the center of the disc; this assertion is exploited in the next result.

Theorem 6. (Liouville). *Let F be a holomorphic function on the complex plane (an entire function) which is also bounded. Then F must be a constant.*

Proof. Assume without loss of generality that $|F|$ is bounded by 1. Fix a point P in the plane. Applying the Cauchy estimates to F on the disc $D(P, R)$ yields that

$$|F'(P)| \leq \frac{1}{R^1}.$$

Letting R tend to $+\infty$ yields that $F'(P) = 0$. Since P was arbitrary, we see that $F' \equiv 0$. A simple calculus exercise now shows that F must be constant. ■

One of the most dramatic applications of Liouville's theorem is in the proof of the fundamental theorem of algebra. That is our next task:

Theorem 7. *Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k$ be a non-constant polynomial. Then there is a point z at which p vanishes.*

Proof. Suppose not. Then $F(z) = 1/p(z)$ is an entire function. Since a non-constant polynomial blows up at infinity, F must be bounded. By Liouville's theorem, F is a constant. Hence p is constant, and therefore has degree zero. This contradiction completes the proof. ■

Let k be the degree of the polynomial p. Notice that if the polynomial p vanishes at the point r_1, then the Euclidean algorithm implies that p is divisible by $(z - r_1)$: that is to say, $p(z) = (z - r_1) \cdot p_1(z)$ for a polynomial p_1 of degree $k - 1$. If $k - 1 \geq 1$, then we may apply the preceding result to p_1. Continuing in this fashion, we obtain that p may be expressed as a product of linear factors:

$$p(z) = (z - r_1) \cdot (z - r_2) \cdots (z - r_k).$$

We conclude this brief overview of elementary complex analysis by recalling the argument principle and Hurwitz's theorem.

Theorem 8. (The Argument Principle). *Let F be holomorphic on a domain U and let γ be a topologically trivial, positively oriented, simple closed curve in U. Assume that F does not vanish on γ. We can be sure, by Theorem 4, that there are at most finitely many, say k, zeros of F inside γ (counting multiplicity). Then we have that*

$$k = \frac{1}{2\pi i} \oint_\gamma \frac{F'(\zeta)}{F(\zeta)}\, d\zeta.$$

Sketch of Proof. By an easy reduction, it is enough to prove the result when $k = 1$. A second reduction allows us to consider the case when γ is a positively oriented circle. After a change of coordinates, let us suppose that F has a simple zero at the point $P = 0$ inside γ. By writing out the power series expansions for F and for F', we find that

$$\frac{F'(\zeta)}{F(\zeta)} = \frac{1}{\zeta} + h(\zeta),$$

where h is holomorphic near 0. Of course, h integrates to 0, by the Cauchy integral theorem. And it is easily calculated that

$$\frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta}\, d\zeta = 1,$$

completing the proof. ■

Let U be a domain and $\{F_j\}$ a sequence of holomorphic functions on U that converges, uniformly on compact sets, to a limit function F. It is an easy consequence of the Cauchy integral formula that the limit function is also holomorphic. Let us now use the argument principle to see how the zeros of F are related to the zeros of the F_j's.

Theorem 9. (Hurwitz's Theorem). *With $\{F_j\}$ and F as above, if the F_j's are all zero-free, then either F is zero-free or F is identically zero.*