CHAPTER 1

THE JACOBSON RADICAL

This chapter has as its major goal the creation of the first steps needed to construct a general structure theory for associative rings. The aim of any structure theory is the description of some general objects in terms of some simpler ones—simpler in some perceptible sense, perhaps in terms of concreteness, perhaps in terms of tractability. Of essential importance, after one has decided upon these simpler objects, is to find a method of passing down to them and to discover how they weave together to yield the general system with which we began.

In carrying out such a program there are many paths one can follow, many classes of candidates for these simpler objects, and one must choose among these for that theory which is most fruitful in producing decisive results. In the case of rings there seems to be no doubt that the fundamental structure theory laid out by Jacobson is the appropriate one. The best proof of this remark is the host of striking theorems which have resulted from the use of these methods.

1. Modules. Essential to everything that we shall discuss—in fact essential in every phase of algebra—is the notion of a module over a ring R or, in short, an R-module. To be absolutely precise we should say a right R-module for we shall allow the elements of R to act on the module from the right. However we shall merely say R-module, understanding by that term a right R-module. Briefly an R-module is a vector space over a ring R; more formally, DEFINITION. *The additive abelian group M is said to be an R-module if there is a mapping from $M \times R$ to M (sending (m, r) to mr) such that:*

1. $m(a+b) = ma + mb$
2. $(m_1 + m_2)a = m_1 a + m_2 a$
3. $(ma)b = m(ab)$

for all $m \in M$ and all $a, b \in R$.

If R should have a unit element, 1, and if $m1 = m$ for all $m \in M$ we then describe M to be a *unitary* R-module.

Note that the definition made above merely says that the ring elements induce endomorphisms on M considered merely as an additive abelian group and that furthermore these endomorphisms induced behave as they should with respect to the addition and multiplication of such endomorphisms. More succinctly put, R is homomorphically imbedded in the ring of all endomorphisms of the additive group of M. For unitary modules we impose the further condition that this imbedding respect the unit element of R, that is, that it correspond to the identity endomorphism.

Mathematics abounds with examples of modules; we shall limit ourselves to two examples for the moment, constructed intrinsically from R itself.

Let R be any ring and let ρ be a right ideal of R. We impose on ρ a natural R-module structure by defining the action of R on ρ to coincide with the product of elements in R. That ρ is an R-module is nothing but a restatement of the fact that it is a right ideal of R.

Using ρ we can construct yet another R-module; let R/ρ be the quotient group of R by ρ considered as additive groups, that is, R/ρ consists of the cosets $x+\rho$ where x ranges over R. Of course R/ρ is not in general a ring—for this to be true ρ would have to be something more, namely a two-sided ideal of R—but it does at least carry the structure of an R-module. We achieve this by defining $(x+\rho)r \equiv xr+\rho$ for all $x+\rho \in R/\rho$ and all $r \in R$. Since ρ is a right ideal of R this definition of the module action makes sense; the verification of the various module axioms is a routine triviality.

Of course vector spaces over fields are examples of modules, in fact of very nice modules. There we have that only the zero element of the field can annihilate a nonzero vector. For a module M over an arbitrary ring R this may be far from true, indeed it is quite possible to have $Mr=(0)$ for some $r \neq 0$ in R. The situation in which this cannot happen is, in some sense, a decent one and we single it out. We say that M is a *faithful* R-module (or that R acts *faithfully* on M) if $Mr=(0)$ forces $r=0$. We now set up a measure of the lack of fidelity of R on M.

DEFINITION. *If M is an R-module then $A(M) = \{x \in R \mid Mx=(0)\}$.*

LEMMA 1.1.1. *$A(M)$ is a two-sided ideal of R. Moreover, M is a faithful $R/A(M)$-module.*

Proof. That $A(M)$ is a right ideal of R is immediate from the axioms for an R-module. To see that it is also a left ideal we proceed as follows: if $r \in R$ and $a \in A(M)$ then $M(ra)=(Mr)a \subset Ma \subset (0)$, hence $ra \in A(M)$. This proves that $A(M)$ is a two-sided ideal of R.

We now make of M an $R/A(M)$-module by defining, for $m \in M$, $r+A(M) \in R/A(M)$, the action $m(r+A(M)) = mr$. If $r+A(M)=r'+A(M)$ then $r-r' \in A(M)$ hence $m(r-r')=0$ for all $m \in M$, that is to say, $mr \equiv mr'$. This in its turn tells us that $m(r+A(M))=mr=mr' = m(r'+A(M))$; the action of $R/A(M)$ on M has been shown to be well defined. The verification that this defines the structure of an $R/A(M)$-module on M we leave to the reader. Finally, to see that M is a faithful $R/A(M)$-module we note that if $m(r+A(M))=0$ for all $m\in M$ then by definition $mr=0$ hence $r\in A(M)$. This says that only the zero element of $R/A(M)$ annihilates all of M.

We formalize some remarks made earlier. Let M be an R-module; for $a\in R$ we define T_a: $M\rightarrow M$ by $mT_a=ma$ for all $m\in M$. Since M is an R-module T_a is an endomorphism of the additive group of M, that is, $(m_1+m_2)T_a=m_1T_a+m_2T_a$ for all $m_1, m_2\in M$. Let $E(M)$ be the set of all endomorphisms of the additive group of M; defining, as usual, for $\phi, \psi\in E(M)$ the sum $\phi+\psi$ by $m(\phi+\psi)=m\phi+m\psi$ and the product $\phi\psi$ by $m(\phi\psi)=(m\phi)\psi$ we see that $E(M)$ is a ring.

Consider the mapping Φ: $R\rightarrow E(M)$ define by $\Phi(a)=T_a$. Going back to the definition of an R-module we see that $\Phi(a+b)=\Phi(a)+\Phi(b)$ and $\Phi(ab)=\Phi(a)\Phi(b)$, in short Φ is a ring homomorphism of R into $E(M)$. What is Ker Φ, the kernel of Φ? Clearly if $a\in A(M)$ then $Ma=(0)$ hence $0=T_a=\Phi(a)$, that is, $a\in$Ker Φ. On the other hand if $a\in$Ker Φ then $T_a=0$ leading to $Ma=MT_a=(0)$, that is, $a\in A(M)$. Therefore the image of R in $E(M)$ is isomorphic to $R/A(M)$. We have proved

LEMMA 1.1.2. *$R/A(M)$ is isomorphic to a subring of $E(M)$.*

In particular if M is a faithful R-module, one for which $A(M)=(0)$, this lemma says that we may consider R as a subring of the ring of endomorphisms of M as an additive group, and so as some ring of endomorphisms of M.

From the interrelation of the R-module M with R we have produced certain elements, the T_a as a ranges over R, in $E(M)$. How do these elements sit in $E(M)$? To be more precise, what elements in $E(M)$ commute with all these T_a?

DEFINITION. *The commuting ring of R on M is* $C(M) = \{\psi \in E(M) \mid T_a\psi = \psi T_a \text{ all } a \in R\}$.

$C(M)$ is certainly a subring of $E(M)$. If $\psi \in C(M)$ then for any $m \in M$ and $a \in R$

$$(m\psi)a = (m\psi)T_a = m(\psi T_a) = m(T_a\psi) = (mT_a)\psi = (ma)\psi,$$

that is, ψ is not only an endomorphism of M as an additive group but is in fact a homomorphism of M into itself as an R-module. We have identified $C(M)$ as the ring of all *module* endomorphisms of M.

Without going into the matter in detail it is clear what one means by a submodule, quotient module, homomorphism of modules. It is equally clear that the usual homomorphism theorems carry over in their entirety from vector spaces to our present context. We single out a special kind of R-module.

DEFINITION. *M is said to be an irreducible R-module if $MR \neq (0)$ and if the only submodules of M are (0) and M.*

For an irreducible R-module M the commuting ring turns out to be rather special. This is the content of an old and basic result known as *Schur's Lemma.*

THEOREM 1.1.1. *If M is an irreducible R-module then $C(M)$ is a division ring.*

Proof. To prove the theorem all we must do is show that any nonzero element in $C(M)$ has an inverse in $C(M)$. Actually we really need but show that if $\theta \neq 0 \in C(M)$ then θ is invertible in $E(M)$. For if $\theta^{-1} \in E(M)$ then from $\theta T_a = T_a\theta$ we immediately have that $T_a\theta^{-1} = \theta^{-1}T_a$, forcing θ^{-1} to be in $C(M)$.

Suppose that $\theta \neq 0 \in C(M)$; if $W = M\theta$ then for all $r \in R$, $Wr = WT_r = (M\theta)T_r = (MT_r)\theta \subset M\theta = W$. Consequently W is a submodule of M. Since $\theta \neq 0$ by the irreducibility of M we deduce that $W\theta = M$ or, in other words, that θ is an onto mapping.

We claim that θ is a monomorphism for, as is quickly verified, Ker θ is a submodule of M and is not all of M since $\theta \neq 0$. Thus Ker $\theta = (0)$. θ being both surjective and a monomorphism we obtain that θ^{-1} exists in $E(M)$. With this Schur's Lemma has been proved.

We pause to look at Schur's Lemma in some very particular contexts.

Let F be a field and let F_n be the ring of all $n \times n$ matrices over F. We consider F_n as the ring of all linear transformations on the vector space V of n-tuples of elements of F. If A is a subset of F_n let $\bar{A}$ be the subalgebra generated by A over F. Clearly V is a faithful F_n, and so, $\bar{A}$-module. V is, in addition, both a unitary and irreducible F_n-module.

We say the set of matrices A is irreducible if V is an irreducible $\bar{A}$-module. In matrix terms this merely says that there is no invertible matrix S in F_n so that

$$S^{-1}aS = \left(\begin{array}{c|c} a_1 & 0 \\ \hline * & a_2 \end{array}\right)$$

for all $a \in A$. The commuting ring of A (that is, of $\bar{A}$) on V is merely the set of all matrices in F_n that commute with all elements of A.

In case F is an algebraically closed field the only division ring having F in its centers and finite dimensional over F, is F itself (we shall see this later). Hence in this particular case the only matrices which commute with an irreducible set of matrices must be the scalars. This is the classical form of Schur's Lemma.

We specialize this discussion to two interesting cases.

1. Let F be the field of real numbers. In F_2 consider the matrix

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

since a has no real characteristic roots we see that $A = \{a\}$ is irreducible. What is the commuting ring of A? From

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

we obtain $\alpha = \delta$, $\beta = -\gamma$ hence the commuting ring of A is the set of all matrices

$$\left\{\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\right\},$$

a field isomorphic to the complex numbers.

2. Again let F be the field of real numbers. In F_4 we consider the two matrices

$$a = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \qquad b = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

We leave it to the reader to verify that $A = \{a, b\}$ is irreducible and that the commuting ring of A is the set of all 4×4 real matrices of the form

$$\left\{\begin{pmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & \delta & -\gamma \\ \gamma & -\delta & \alpha & \beta \\ \delta & \gamma & -\beta & \alpha \end{pmatrix}\right\}.$$

This is a 4-dimensional division ring over the real field which is isomorphic to the real quaternions.

We close this section with an intrinsic description of all the irreducible modules of a given ring R.

LEMMA 1.1.3. *If M is an irreducible R-module then M is isomorphic as a module to R/ρ for some maximal right ideal ρ of R. Moreover there is an $a \in R$ such that $x\text{-}ax \in \rho$ for all $x \in R$. Conversely, for every such maximal right ideal ρ of R, R/ρ is an irreducible R-module.*

Proof. Since M is irreducible, by the very definition we must have that $MR \neq (0)$. Since $S = \{u \in M \mid uR = (0)\}$ is a submodule of M and is not M it must be (0). Equivalently, if $m \neq 0$ is in M then $mR \neq (0)$. However, mR is a submodule of M hence it must be all of M. Define $\Psi: R \rightarrow M$ by $\Psi(r) = mr$ for every $r \in R$. We see at once that Ψ is a homomorphism of R into M as R-modules; since $mR = M$ we have that Ψ is surjective. Finally, $\text{Ker}\,\Psi = \{x \in R \mid mx = 0\}$ is a right ideal ρ; by a standard homomorphism theorem we have that M is isomorphic to R/ρ as an R-module.

Any right ideal of R which properly contains ρ maps, under ϕ, into a submodule of M. Hence ρ is a maximal right ideal of R. We now produce the desired element a in R. Since $mR = M$ there is an element $a \in R$ such that $ma = m$. Therefore for any $x \in R$ $max = mx$, which is to say $m(x - ax) = 0$. This puts $x - ax$ in ρ.

We leave the proof of the converse to the reader.

2. The radical of a ring. In setting up a structure theory for a category of algebraic objects it is desirable to be able to recognize what special classes of objects are "nice" and to be able to measure the lack of "niceness" in the general object of this category. One then wants some sort of passage from the general object to these better-behaved ones.

It is towards this goal that we introduce the radical of a ring. We shall see that a ring having (0) as radical has a rather concrete description in terms of particular and often more manageable rings. Furthermore, any ring modulo its radical will turn out to be a ring having (0) as radical. In order to study a general ring, then, we want to slice out of the ring a certain piece—the so-called radical—in such a way that we do not slice out too much, so that the piece being cut away is capable of description yet at the same time we do not want to cut out too little, so that the object resulting after the excision is also capable of description.

The motivation for the definition we make comes primarily from the representation theory of groups. In the classical theory of finite dimensional algebra the radical was defined in a completely different way. As we shall later see our definition will coincide with the classical one in the classical situations.

DEFINITION. *The radical of R, written as $J(R)$, is the set of all elements of R which annihilate all the irreducible R-modules. If R has no irreducible modules we put $J(R)=R$.*

There are many possible radicals; the one we defined and shall use throughout is often called the *Jacobson radical.*

Note that $J(R)=\bigcap A(M)$ where this intersection runs over all irreducible R-modules M. Since the $A(M)$ are two-sided ideals of R we see that $J(R)$ is a two-sided ideal of R. In all fairness we should call $J(R)$ the right radical of R for it has been defined in terms of right R-modules. We could similarly define a left radical. Fortunately these two turn out to be the same so we shall be spared making such a left-right distinction.

In Lemma 1.1.3 we saw that every irreducible R-module arises as R/ρ where ρ is a maximal right ideal enjoying one further property, namely, the existence of some element $a \in R$ such that $x\text{-}ax \in \rho$ for all $x \in R$. This motivates the

DEFINITION. *A right ideal ρ of R is said to be regular if there is an $a \in R$ such that $x - ax \in \rho$ for all $x \in R$.*

If R has a unit element (in fact, a left unit will do) then all its right ideals are regular.

DEFINITION. *If ρ is a right ideal of R then $(\rho: R) = \{x \in R \mid Rx \subset \rho\}$.*

Let ρ be a maximal right ideal of R which is also assumed to be regular and let $M = R/\rho$. What is $A(M)$? If $x \in A(M)$ then $Mx = (0)$, which is to say, $(r+\rho)x = \rho$ for all $r \in R$. This latter says that $Rx \subset \rho$, hence $A(M) \subset (\rho: R)$. Similarly $(\rho: R) \subset A(M)$ whence $A(M) = (\rho: R)$. Since ρ is regular there is an $a \in R$ with $x - ax \in \rho$ for all $x \in R$; in particular, if $x \in (\rho: R)$ then since $ax \in Rx \subset \rho$ we get $x \in \rho$. We thus see that $A(M) = (\rho: R)$ is the largest two-sided ideal of R which lies in ρ. In view of Lemma 1.1.3 we can say

THEOREM 1.2.1. *$J(R) = \bigcap (\rho: R)$ where ρ runs over all the regular maximal right ideals of R, and where $(\rho: R)$ is the largest two-sided ideal of R lying in ρ.*

We want to sharpen this. To do so we need a preliminary result.