1
Linear Analysis

This chapter introduces theoretical tools for studying stochastic differential equations in later chapters. §1.1 and §1.2 review Banach and Hilbert spaces, the mathematical structures given to sets of random variables and the natural home for solutions of differential equations. §1.3 reviews the theory of linear operators, especially the spectral theory of compact and symmetric operators, and §1.4 reviews Fourier analysis.

1.1 Banach spaces C^r and L^p

Banach and Hilbert spaces are fundamental to the analysis of differential equations and random processes. This section treats Banach spaces, reviewing first the notions of norm, convergence, and completeness before giving Definition 1.7 of a Banach space. We assume readers are familiar with real and complex vector spaces.

Definition 1.1 (norm) A *norm* $\|\cdot\|$ is a function from a real (respectively, complex) vector space X to $\mathbb{R}^+$ such that

(i) $\|u\| = 0$ if and only if $u = 0$,
(ii) $\|\lambda u\| = |\lambda|\,\|u\|$ for all $u \in X$ and $\lambda \in \mathbb{R}$ (resp., $\mathbb{C}$), and
(iii) $\|u + v\| \le \|u\| + \|v\|$ for all $u, v \in X$ (triangle inequality).

A *normed vector space* $(X, \|\cdot\|)$ is a vector space X with a norm $\|\cdot\|$. If only conditions (ii) and (iii) hold, $\|\cdot\|$ is called a *semi-norm* and denoted $|\cdot|_X$.

Example 1.2 $(\mathbb{R}^d, \|\cdot\|_2)$ is a normed vector space with

$$\|\boldsymbol{u}\|_2 := \left(|u_1|^2 + \cdots + |u_d|^2\right)^{1/2}$$

for the column vector $\boldsymbol{u} = [u_1, \ldots, u_d]^{\mathsf{T}} \in \mathbb{R}^d$, where $|\cdot|$ denotes absolute value. More generally, $\|\boldsymbol{u}\|_\infty := \max\{|u_1|, \ldots, |u_d|\}$ and $\|\boldsymbol{u}\|_p := (|u_1|^p + \cdots + |u_d|^p)^{1/p}$ for $p \ge 1$ is a norm and $(\mathbb{R}^d, \|\cdot\|_p)$ is a normed vector space. When $d = 1$, these norms are all equal to the absolute value.

Definition 1.3 (domain) A domain D is a non-trivial, connected, open subset of $\mathbb{R}^d$ and a domain is bounded if $D \subset \left\{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_2 \le R\right\}$ for some $R > 0$. The boundary of a domain is denoted ∂D and we always assume the boundary is piecewise smooth (e.g., the boundary of a polygon or a sphere).

Example 1.4 (continuous functions) For a subset $D \subset \mathbb{R}^d$, let $\mathrm{C}(D)$ denote the set of real-valued continuous functions on D. If D is a domain, functions in $\mathrm{C}(D)$ may be unbounded. However, functions in $\mathrm{C}(\bar{D})$, where $\bar{D}$ is the closure of D, are bounded and $\big(\mathrm{C}(\bar{D}), \|\cdot\|_\infty\big)$ is a normed vector space with the supremum norm,

$$\|u\|_\infty := \sup_{\boldsymbol{x} \in \bar{D}} |u(\boldsymbol{x})|, \qquad u \in \mathrm{C}(\bar{D}).$$

A norm $\|\cdot\|$ on a vector space X measures the size of elements in X and provides a notion of convergence: for $u, u_n \in X$, we write $u = \lim_{n\to\infty} u_n$ or $u_n \to u$ as $n \to \infty$ in $(X, \|\cdot\|)$ if $\|u_n - u\| \to 0$ as $n \to \infty$. For example, the notion of convergence on $\mathrm{C}(\bar{D})$ is known as uniform convergence.

Definition 1.5 (uniform and pointwise convergence) We say $u_n \in \mathrm{C}(\bar{D})$ converges *uniformly* to a limit u if $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Explicitly, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $\boldsymbol{x} \in \bar{D}$ and all $n \geq N$, $|u_n(\boldsymbol{x}) - u(\boldsymbol{x})| < \epsilon$. In uniform convergence, N depends only on ϵ. This should be contrasted with the notion of pointwise convergence, which applies to all functions $u_n : D \to \mathbb{R}$. We say $u_n \to u$ *pointwise* if, for every $\boldsymbol{x} \in D$ and every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|u_n(\boldsymbol{x}) - u(\boldsymbol{x})| < \epsilon$. In pointwise convergence, N may depend both on ϵ and $\boldsymbol{x}$.

There are many techniques, both computational and analytical, for finding approximate solutions $u_n \in X$ to mathematical problems posed on a vector space X. When u_n is a Cauchy sequence and X is complete, u_n converges to some $u \in X$, the so-called limit point, and this is often key in showing a mathematical model is well posed and proving the existence of a solution.

Definition 1.6 (Cauchy sequence, complete) Consider a normed vector space $(X, \|\cdot\|)$. A sequence $u_n \in X$ for $n \in \mathbb{N}$ is called a *Cauchy sequence* if, for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\|u_n - u_m\| < \epsilon \qquad \text{for all } n, m \geq N.$$

A normed vector space $(X, \|\cdot\|)$ is said to be *complete* if every Cauchy sequence u_n in X converges to a limit point $u \in X$. In other words, there exists a $u \in X$ such that $\|u_n - u\| \to 0$ as $n \to \infty$.

Definition 1.7 (Banach space) A *Banach space* is a complete normed vector space.

Example 1.8 $(\mathbb{R}, |\cdot|)$ and $(\mathbb{R}^d, \|\cdot\|_p)$ for $1 \leq p \leq \infty$ are Banach spaces.

Example 1.9 $\big(\mathrm{C}(\bar{D}), \|\cdot\|_\infty\big)$ is a Banach space if D is bounded. See Exercise 1.1. If D is unbounded, the set of bounded continuous functions $\mathrm{C}_\mathrm{b}(D)$ on D gives a Banach space.

The contraction mapping theorem is used in Chapters 3, 8, and 10 to prove the existence and uniqueness of solutions to initial-value problems.

Theorem 1.10 (contraction mapping) *Let Y be a non-empty closed subset of the Banach space $(X, \|\cdot\|)$. Consider a mapping $\mathcal{J} : Y \to Y$ such that, for some $\mu \in (0, 1)$,*

$$\|\mathcal{J}u - \mathcal{J}v\| \leq \mu \|u - v\|, \qquad \text{for all } u, v \in Y. \tag{1.1}$$

There exists a unique fixed point of $\mathcal{J}$ in Y; that is, there is a unique $u \in Y$ such that $\mathcal{J}u = u$.

Proof Fix $u_0 \in Y$ and consider $u_n = \mathcal{J}^n u_0$ (the nth iterate of u_0 under application of $\mathcal{J}$). The sequence u_n is easily shown to be Cauchy in Y using (1.1) and therefore converges to a limit $u \in Y$ because Y is complete (as a closed subset of X). Now $u_n \to u$ and hence $u_{n+1} = \mathcal{J} u_n \to \mathcal{J} u$ as $n \to \infty$. We conclude that u_n converges to a fixed point of $\mathcal{J}$.

If $u, v \in Y$ are both fixed points of $\mathcal{J}$, then $\mathcal{J}u - \mathcal{J}v = u - v$. But (1.1) holds and hence $u = v$ and the fixed point is unique. □

Spaces of continuously differentiable functions

The smoothness, also called regularity, of a function is described by its derivatives and we now define spaces of functions with a given number of continuous derivatives. For a domain $D \subset \mathbb{R}^d$ and Banach space $(Y, \|\cdot\|_Y)$, consider a function $u \colon D \to Y$. We denote the partial derivative operator with respect to x_j by $\mathcal{D}_j := \frac{\partial}{\partial x_j}$. Given a multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$, we define $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_d$ and $\mathcal{D}^{\boldsymbol{\alpha}} := \mathcal{D}_1^{\alpha_1} \cdots \mathcal{D}_d^{\alpha_d}$, so that

$$\mathcal{D}^{\boldsymbol{\alpha}} u = \frac{\partial^{|\boldsymbol{\alpha}|} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

Definition 1.11 (continuous functions)

(i) $\mathrm{C}(D, Y)$ is the set of continuous functions $u \colon D \to Y$. If D is bounded, we equip $\mathrm{C}(\bar{D}, Y)$ with the norm

$$\|u\|_\infty := \sup_{\boldsymbol{x} \in \bar{D}} \|u(\boldsymbol{x})\|_Y. \tag{1.2}$$

(ii) $\mathrm{C}^r(D, Y)$ with $r \in \mathbb{N}$ is the set of functions $u \colon D \to Y$ such that $\mathcal{D}^{\boldsymbol{\alpha}} u \in \mathrm{C}(D, Y)$ for $|\boldsymbol{\alpha}| \le r$; that is, functions whose derivatives up to and including order r are continuous. We equip $\mathrm{C}^r(\bar{D}, Y)$ with the norm

$$\|u\|_{\mathrm{C}^r(\bar{D}, Y)} := \sum_{0 \le |\boldsymbol{\alpha}| \le r} \|\mathcal{D}^{\boldsymbol{\alpha}} u\|_\infty.$$

We abbreviate the notation so that $\mathrm{C}(D, \mathbb{R})$ is denoted by $\mathrm{C}(D)$ and $\mathrm{C}^r(D, \mathbb{R})$ by $\mathrm{C}^r(D)$.

Proposition 1.12 *If the domain D is bounded,* $\mathrm{C}(\bar{D}, Y)$ *and* $\mathrm{C}^r(\bar{D}, Y)$ *are Banach spaces.*

Proof The case of $\mathrm{C}(\bar{D})$ is considered in Exercise 1.1. □

The following sets of continuous functions, which are not provided with a norm, are also useful.

Definition 1.13 (infinitely differentiable functions)

(i) $\mathrm{C}^\infty(D, Y)$ is the set $\bigcap_{r \in \mathbb{N}} \mathrm{C}^r(D, Y)$ of infinitely differentiable functions from D to Y.

(ii) $\mathrm{C}_c^\infty(D, Y)$ is the set of $u \in \mathrm{C}^\infty(D, Y)$ such that $\operatorname{supp} u$ is a compact subset of D, where the support $\operatorname{supp} u$ denotes the closure of $\{\boldsymbol{x} \in D \colon u(\boldsymbol{x}) \neq 0\}$. (The definition of compact is recalled in Definition 1.66).

The spaces $C^r(D,Y)$ specify the regularity of a function via the number, r, of continuous derivatives. More refined concepts of regularity include Hölder and Lipschitz regularity.

Definition 1.14 (Hölder and Lipschitz) Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces. A function $u\colon X \to Y$ is *Hölder continuous* with constant $\gamma \in (0,1]$ if there is a constant $L > 0$ so that

$$\|u(x_1) - u(x_2)\|_Y \le L\|x_1 - x_2\|_X^\gamma, \qquad \forall x_1, x_2 \in X.$$

If the above holds with $\gamma = 1$, then u is *Lipschitz continuous* or *globally* Lipschitz continuous to stress that L is uniform for $x_1, x_2 \in X$. The space $C^{r,\gamma}(D)$ is the set of functions in $C^r(D)$ whose rth derivatives are Hölder continuous with exponent γ.

Lebesgue integrals and measurability

The Lebesgue integral is an important generalisation of the Riemann integral. For a function $u\colon [a,b] \to \mathbb{R}$, the Riemann integral $\int_a^b u(x)\,dx$ is given by a limit of sums $\sum_{j=0}^{N-1} u(\xi_j)(x_{j+1} - x_j)$ for points $\xi_j \in [x_j, x_{j+1}]$, with respect to refinement of the partition $a = x_0 < \cdots < x_N = b$. In other words, u is approximated by a piecewise constant function, whose integral is easy to evaluate, and a limiting process defines the integral of u. The Lebesgue integral is also defined by a limit, but instead of piecewise constant approximations on a partition of $[a,b]$, approximations constant on *measurable sets* are used.

Let 1_F be the indicator function of a set F so

$$1_F(x) := \begin{cases} 1, & x \in F, \\ 0, & x \notin F. \end{cases}$$

Suppose that $\{F_j\}$ are measurable sets in $[a,b]$ (see Definition 1.15) and $\mu(F_j)$ denotes the measure of F_j (e.g., if $F_j = [a,b]$ then $\mu([a,b]) = |b-a|$). The Lebesgue integral of u with respect to the measure μ is defined via

$$\int_a^b u(x)\,d\mu(x) = \lim \sum_j u_j\,\mu(F_j),$$

where the limit is taken as the functions $\sum_j u_j 1_{F_j}(x)$ converge to $u(x)$. The idea is illustrated in Figure 1.1, where the function $u(x)$ is approximated by $\sum_{i=1}^3 u_j 1_{F_j}(x)$ for

$$F_1 = u^{-1}([-1,-0.5]), \qquad F_2 = u^{-1}((-0.5,0.5]), \qquad F_3 = u^{-1}((0.5,1]),$$
$$u_1 = -0.8, \qquad u_2 = 0, \qquad u_3 = 0.8.$$

Here, $u^{-1}([a,b]) := \{x \in \mathbb{R} : u(x) \in [a,b]\}$. To precisely define the Lebesgue integral, we must first form a collection of subsets $\mathcal{F}$ that we can measure.

Definition 1.15 (σ-algebra) A set $\mathcal{F}$ of subsets of a set X is a *σ-algebra* if

(i) the empty set $\{\} \in \mathcal{F}$,
(ii) the complement $F^c := \{x \in X : x \notin F\} \in \mathcal{F}$ for all $F \in \mathcal{F}$, and
(iii) the union $\cup_{j\in\mathbb{N}} F_j \in \mathcal{F}$ for $F_j \in \mathcal{F}$.

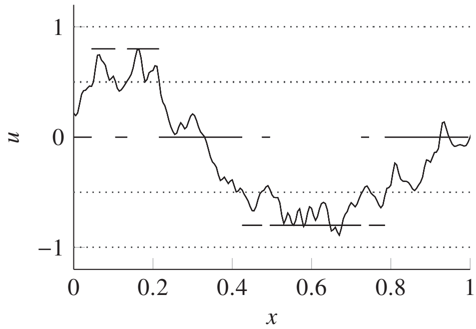

Figure 1.1 We approximate a function $u(x)$ by the simple function $\sum_{j=1}^{3} u_j 1_{F_j}(x)$. The sets F_j arise by partitioning the y-axis and using the pullback sets, which are measurable when u is measurable. In contrast to the Riemann integral, F_j are not simple intervals.

Thus, a σ-algebra is a collection of subsets of X that contains the empty set and is closed under forming complements and countable unions. Any $F \in \mathcal{F}$ is known as a *measurable set* and the pair $(X, \mathcal{F})$ is known as a *measurable space*.

It is natural to ask why the power set of X (i.e., the set of all subsets of X) is not chosen for $\mathcal{F}$. It turns out, for example using Vitali sets or in the Banach–Tarski paradox, that non-intuitive effects arise through measuring every set.

For topological spaces like $\mathbb{R}^d$ or Banach spaces, $\mathcal{F}$ is chosen to be the smallest σ-algebra that contains all open sets; this is known as the *Borel σ-algebra*.

Definition 1.16 (Borel σ-algebra) For a topological space Y, $\mathcal{B}(Y)$ denotes the Borel σ-algebra and equals the smallest σ-algebra containing all open subsets of Y.

Definition 1.17 (measure) A measure μ on a measurable space $(X, \mathcal{F})$ is a mapping from $\mathcal{F}$ to $\mathbb{R}^+ \cup \{\infty\}$ such that

(i) the empty set has measure zero, $\mu(\{\}) = 0$, and
(ii) $\mu\big(\cup_{j\in\mathbb{N}} F_j\big) = \sum_{j\in\mathbb{N}} \mu(F_j)$ if $F_j \in \mathcal{F}$ are disjoint (i.e., $F_k \cap F_j = \{\}$ for $k \neq j$).

Together $(X, \mathcal{F}, \mu)$ form a *measure space*. We say the measure space is *σ-finite* if $X = \cup_{j=1}^{\infty} F_j$ for some $F_j \in \mathcal{F}$ with $\mu(F_j) < \infty$. A set $F \in \mathcal{F}$ such that $\mu(F) = 0$ is known as a *null set* and the measure space is said to be *complete* if all subsets of null sets belong to $\mathcal{F}$.

Any measure space $(X, \mathcal{F}, \mu)$ can be extended to a complete measure space by adding in all subsets of null sets. In this book, we always implicitly assume this extension to a complete measure space has been made.

Example 1.18 (Lebesgue measure) The usual notion of volume of subsets of $\mathbb{R}^d$ gives rise to Lebesgue measure, which we denote Leb, on the Borel sets $\mathcal{B}(\mathbb{R}^d)$. The measure space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \text{Leb})$ is σ-finite and (once subsets of null sets are included) complete.

Integrals are defined for measurable functions. Let $(X, \mathcal{F}, \mu)$ be a measure space and $(Y, \|\cdot\|_Y)$ be a Banach space.

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Definition 1.19 (measurable) (i) A function $u\colon X \to \mathbb{R}$ is *$\mathcal{F}$-measurable* if $\{x \in X : u(x) \le a\} \in \mathcal{F}$ for every $a \in \mathbb{R}$. A function $u\colon X \to Y$ is *$\mathcal{F}$-measurable* if the pullback set $u^{-1}(G) \in \mathcal{F}$ for any $G \in \mathcal{B}(Y)$. We simply write *u is measurable* if the underlying σ-algebra is clear and write *u is Borel measurable* if $\mathcal{F}$ is a Borel σ-algebra.

(ii) A measurable function $u = 0$ *almost surely* (a.s.) or *almost everywhere* (a.e.) if $\mu(\{x \in X : u(x) \neq 0\}) = 0$. We sometimes write μ-a.s. to stress the measure.

The following lemma requires the underlying measure space $(X, \mathcal{F}, \mathbb{P})$ to be *complete* (as we assume throughout the book). See Exercise 1.2.

Lemma 1.20 *If $u_n : X \to Y$ is measurable and $u_n(x) \to u(x)$ for almost all $x \in X$, then $u\colon X \to Y$ is measurable.*

Finally, we define the integral of a measurable function $u\colon X \to Y$ with respect to $(X, \mathcal{F}, \mu)$.

Definition 1.21 (integral)

(i) A function $s\colon X \to Y$ is *simple* if there exist $s_j \in Y$ and $F_j \in \mathcal{F}$ for $j = 1, \ldots, N$ with $\mu(F_j) < \infty$ such that

$$s(x) = \sum_{j=1}^{N} s_j 1_{F_j}(x), \qquad x \in X.$$

The integral of a simple function s with respect to a measure space $(X, \mathcal{F}, \mu)$ is

$$\int_X s(x)\, d\mu(x) := \sum_{j=1}^{N} s_j\, \mu(F_j). \tag{1.3}$$

(ii) We say a measurable function u is *integrable with respect to μ* if there exist simple functions u_n such that $u_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in X$ and u_n is a Cauchy sequence in the sense that, for all $\epsilon > 0$,

$$\int_X \|u_n(x) - u_m(x)\|_Y \, d\mu(x) < \epsilon,$$

for any n, m sufficiently large. Notice that $s(x) = \|u_n(x) - u_m(x)\|_Y$ is a simple function from X to $\mathbb{R}$ and the integral is defined by (1.3).

(iii) If u is integrable, define

$$\int_X u(x)\, d\mu(x) := \lim_{n\to\infty} \int_X u_n(x)\, d\mu(x).$$

Note that $\int_X \|u(x)\|_Y \, d\mu(x) < \infty$ (see Exercise 1.3). If $F \in \mathcal{F}$, define

$$\int_F u(x)\, d\mu(x) := \int_X u(x) 1_F(x)\, d\mu(x).$$

When $Y = \mathbb{R}^d$, the above definition gives the *Lebesgue integral* and, for a Borel set $D \subset \mathbb{R}^d$, the integral with respect to $(D, \mathcal{B}(D), \text{Leb})$ is denoted

$$\int_D u(\boldsymbol{x})\, d\text{Leb}(\boldsymbol{x}) = \int_D u(\boldsymbol{x})\, d\boldsymbol{x}.$$

This corresponds to the usual notion of volume beneath a surface. However, the Lebesgue integral is much more general than the Riemann integral, as the partitions essential for the definition of the Riemann integral do not make sense for the probability spaces of Chapter 4. Definition 1.21 includes the case where Y is a Banach space, the so-called *Bochner integral*.

The following elementary properties hold in general for integrable functions.

(i) For integrable functions $u, v\colon X \to Y$ and $a, b \in \mathbb{R}$,

$$\int_X \big(au(x) + bv(x)\big)\, d\mu(x) = a \int_X u(x)\, d\mu(x) + b \int_X v(x)\, d\mu(x).$$

(ii) For $u\colon X \to Y$,

$$\left\| \int_X u(x)\, d\mu(x) \right\|_Y \le \int_X \|u(x)\|_Y\, d\mu(x). \tag{1.4}$$

We state without proof the dominated convergence theorem on limits of integrals and Fubini's theorem for integration on product spaces.

Theorem 1.22 (dominated convergence) *Consider a sequence of measurable functions* $u_n\colon X \to Y$ *such that* $u_n(x) \to u(x)$ *in* Y *as* $n \to \infty$ *for almost all* $x \in X$. *If there is a real-valued integrable function* $\bar{U}$ *such that* $\|u_n(x)\|_Y \le |\bar{U}(x)|$ *for* $n \in \mathbb{N}$ *and almost all* $x \in X$, *then*

$$\lim_{n\to\infty} \int_X u_n(x)\, d\mu(x) = \int_X \lim_{n\to\infty} u_n(x)\, d\mu(x) = \int_X u(x)\, d\mu(x).$$

Now suppose that $(X_k, \mathcal{F}_k, \mu_k)$ for $k = 1, 2$ are a pair of σ-finite measure spaces.

Definition 1.23 (product measure) Denote by $\mathcal{F}_1 \times \mathcal{F}_2$ the smallest σ-algebra containing all sets $F_1 \times F_2$ for $F_k \in \mathcal{F}_k$. The *product measure* $\mu_1 \times \mu_2$ on $\mathcal{F}_1 \times \mathcal{F}_2$ is defined by $(\mu_1 \times \mu_2)(F_1 \times F_2) \coloneqq \mu_1(F_1)\, \mu_2(F_2)$.

Theorem 1.24 (Fubini) *Suppose that* $(X_k, \mathcal{F}_k, \mu_k)$ *for* $k = 1, 2$ *are* σ*-finite measure spaces and consider a measurable function* $u\colon X_1 \times X_2 \to Y$. *If*

$$\int_{X_2} \left(\int_{X_1} \|u(x_1, x_2)\|_Y\, d\mu_1(x_1) \right) d\mu_2(x_2) < \infty, \tag{1.5}$$

then u *is integrable with respect to the product measure* $\mu_1 \times \mu_2$ *and*

$$\begin{aligned}\int_{X_1\times X_2} u(x_1, x_2)\, d(\mu_1 \times \mu_2)(x_1, x_2) &= \int_{X_2} \left(\int_{X_1} u(x_1, x_2)\, d\mu_1(x_1) \right) d\mu_2(x_2)\\ &= \int_{X_1} \left(\int_{X_2} u(x_1, x_2)\, d\mu_2(x_2) \right) d\mu_1(x_1).\end{aligned}$$

To understand the necessity of (1.5), see Exercise 1.7.

Example 1.25 For Borel sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, both measure spaces $(X, \mathcal{B}(X), \mathrm{Leb})$ and $(Y, \mathcal{B}(Y), \mathrm{Leb})$ are σ-finite. Consider $u\colon X \times Y \to \mathbb{R}$ such that $\int_X \int_Y |u(\boldsymbol{x}, \boldsymbol{y})|\, d\boldsymbol{y}\, d\boldsymbol{x} < \infty$. Then Fubini's theorem applies and

$$\int_{X\times Y} u(\boldsymbol{x}, \boldsymbol{y})\, d(\mathrm{Leb} \times \mathrm{Leb})(\boldsymbol{x}, \boldsymbol{y}) = \int_X \int_Y u(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y}\, d\boldsymbol{x} = \int_Y \int_X u(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{x}\, d\boldsymbol{y}.$$

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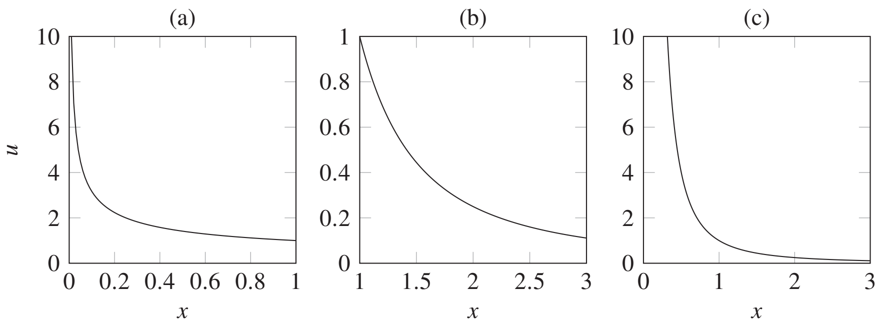

Figure 1.2 (a) If $u(x) = x^{-1/2}$ then $u \in L^p(0,1)$ for $1 \le p < 2$; (b) if $u(x) = x^{-2}$ then $u \in L^p(1,\infty)$ for $p > 2$; (c) if $u(x) = x^{-2}$ then $u \notin L^p(0,\infty)$ for any $p \ge 1$.

L^p spaces

With Lebesgue integrals now defined, we introduce spaces of functions that have finite integrals.

Definition 1.26 (L^p spaces) Let $(Y, \|\cdot\|_Y)$ be a Banach space and $1 \le p < \infty$.

(i) For a domain D, $L^p(D)$ is the set of Borel measurable functions $u\colon D \to \mathbb{R}$ with $\|u\|_{L^p(D)} < \infty$, where

$$\|u\|_{L^p(D)} := \left(\int_D |u(\boldsymbol{x})|^p \, d\boldsymbol{x} \right)^{1/p}. \tag{1.6}$$

$L^1(D)$ comprises the real-valued functions on D that are integrable with respect to Lebesgue measure.

(ii) Let $(X, \mathcal{F}, \mu)$ be a measure space. The space $L^p(X,Y)$ is the set of $\mathcal{F}$-measurable functions $u\colon X \to Y$ such that $\|u\|_{L^p(X,Y)} < \infty$, where

$$\|u\|_{L^p(X,Y)} := \left(\int_X \|u(x)\|_Y^p \, d\mu(x) \right)^{1/p}.$$

$L^1(X,Y)$ comprises the integrable (with respect to μ) functions $X \to Y$. We write $L^p(X)$ for $L^p(X,\mathbb{R})$.

(iii) $L^\infty(X,Y)$ is the set of $\mathcal{F}$-measurable functions $u\colon X \to Y$ such that

$$\|u\|_{L^\infty(X,Y)} := \operatorname*{ess\,sup}_{x \in X} \|u(x)\|_Y < \infty$$

and $\operatorname{ess\,sup}_{x\in X} \|u(x)\|_Y$ denotes the essential supremum or the smallest number that bounds $\|u\|_Y$ almost everywhere. We write $L^\infty(X)$ for $L^\infty(X,\mathbb{R})$.

We examine further the case $p = 2$ in §1.2. The following example illustrates $L^p(D)$ for different choices of p and D.

Example 1.27 Let $D = (0,1)$ and $u(x) = x^\alpha$ with $\alpha < 0$. Then $u \in L^p(D)$ if and only if

$$\int_0^1 x^{\alpha p} \, dx = \left[\frac{x^{1+\alpha p}}{1+\alpha p} \right]_0^1 < \infty \iff 1 + \alpha p > 0.$$

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Hence, $u \in L^p(D)$ for $1 \le p < -1/\alpha$. See Figure 1.2(a) for the case $\alpha = -1/2$.

If $D = (1,\infty)$, then $u \in L^p(D)$ if and only if $1 + \alpha p < 0$ or $p > -1/\alpha$. See Figure 1.2(b) for the case $\alpha = -2$. If $D = (0,\infty)$ then $u \notin L^p(D)$ for any $p \ge 1$, as u grows too quickly near the origin or decays too slowly at infinity; see Figure 1.2(c).

More generally, it can be shown $L^q(D) \subset L^p(D)$ for $1 \le p \le q$ for any bounded domain D. See Exercise 1.9.

The basic inequalities for working on Lebesgue spaces are the following:

Hölder's inequality: Suppose that $1 \le p,q \le \infty$ with $1/p + 1/q = 1$ (p,q are said to be *conjugate exponents* and this includes $\{p,q\} = \{1,\infty\}$). For $u \in L^p(X)$ and $v \in L^q(X)$,

$$\|uv\|_{L^1(X)} \le \|u\|_{L^p(X)}\|v\|_{L^q(X)}. \tag{1.7}$$

Minkowski's inequality: For $u,v \in L^p(X,Y)$,

$$\|u+v\|_{L^p(X,Y)} \le \|u\|_{L^p(X,Y)} + \|v\|_{L^p(X,Y)}.$$

Minkowski's inequality provides the triangle inequality needed to establish that $\|\cdot\|_{L^p(X,Y)}$ is a norm. Note however that, as functions in $L^p(X,Y)$ in Definition 1.26 are unique only up to sets of measure zero, $\|\cdot\|_{L^p(X,Y)}$ fails the first axiom in Definition 1.1. The trick is to work on the set of equivalence classes of functions that are equal almost everywhere with respect to the given measure on X. Then, when we refer to functions in the $L^p(X,Y)$ spaces, we mean functions that are representative of those equivalence classes. Using a rigorous definition, it can be shown that $L^p(X,Y)$ is a complete normed vector space.

Lemma 1.28 *$L^p(X,Y)$ is a Banach space for $1 \le p \le \infty$.*

1.2 Hilbert spaces L^2 and H^r

Hilbert spaces are Banach spaces with the additional structure of an inner product.

Definition 1.29 (inner product) An *inner product* on a real (resp., complex) vector space X is a function $\langle\cdot,\cdot\rangle\colon X \times X \to \mathbb{R}$ (resp., $\mathbb{C}$) that is

(i) positive definite: $\langle u,u\rangle \ge 0$ and $\langle u,u\rangle = 0$ if and only if $u = 0$,
(ii) conjugate symmetric: $\langle u,v\rangle = \overline{\langle v,u\rangle}$, where $\overline{}$ denotes the complex conjugate of u, and
(iii) linear in the first argument: $\langle \lambda u + \mu v, w\rangle = \lambda\langle u,w\rangle + \mu\langle v,w\rangle$ for real (resp., complex) λ,μ and $u,v,w \in X$.

Definition 1.30 (Hilbert space) Let H be a real (resp., complex) vector space with inner product $\langle\cdot,\cdot\rangle$. Then, H is a real (resp., complex) *Hilbert space* if it is complete with respect to the induced norm $\|u\| \coloneqq \langle u,u\rangle^{1/2}$. In particular, any Hilbert space is a Banach space.

Example 1.31 $\mathbb{R}^d$ and $\mathbb{C}^d$ are Hilbert spaces with the inner product

$$\langle \boldsymbol{x},\boldsymbol{y}\rangle \coloneqq \boldsymbol{x}^{\mathsf{T}}\bar{\boldsymbol{y}} = x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_d\bar{y}_d.$$

In this book, the norm $\|\cdot\|_2$ on $\mathbb{R}^d$ denotes $\|\boldsymbol{x}\|_2 \coloneqq \langle\boldsymbol{x},\boldsymbol{x}\rangle^{1/2} = \left(x_1^2 + \cdots + x_d^2\right)^{1/2}$.

The following inequality is frequently used when working with Hilbert spaces.

Lemma 1.32 (Cauchy–Schwarz inequality) *Let H be a Hilbert space. Then*

$$\left|\langle u,v\rangle\right| \le \|u\|\,\|v\|, \qquad \forall u,v \in H. \tag{1.8}$$

Proof See Exercise 1.10. □

Hilbert space $L^2(D)$

Recall the space $L^p(D)$ introduced in Definition 1.26. When $p = 2$, there is an inner product and $L^2(D)$ is a Hilbert space. Functions in this space are called *square integrable*.

Proposition 1.33 *For any domain D, $L^2(D)$ is a real Hilbert space with inner product*

$$\langle u,v\rangle_{L^2(D)} := \int_D u(\boldsymbol{x})v(\boldsymbol{x})\,d\boldsymbol{x}.$$

Proof $L^2(D)$ is a Banach space by Lemma 1.28. It is elementary to verify that $\langle\cdot,\cdot\rangle_{L^2(D)}$ is an inner product and that $\langle u,u\rangle_{L^2(D)}^{1/2}$ agrees with (1.6) with $p = 2$. □

More generally, we may work with the L^2 space of functions taking values in any Hilbert space H.

Proposition 1.34 *Let $(X,\mathcal{F},\mu)$ be a measure space and H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Then, $L^2(X,H)$ is a Hilbert space with inner product*

$$\langle u,v\rangle_{L^2(X,H)} := \int_X \langle u(x),v(x)\rangle\,d\mu(x).$$

Example 1.35 Later on, we meet the complex Hilbert space $L^2(\mathbb{R}^d,\mathbb{C})$ with inner product $\langle u,v\rangle_{L^2(\mathbb{R}^d,\mathbb{C})} := \int_{\mathbb{R}^d} u(\boldsymbol{x})\overline{v(\boldsymbol{x})}\,d\boldsymbol{x}$. For stochastic PDEs and a second measurable space $(\Omega,\mathcal{F},\mathbb{P})$, we use $L^2(\Omega,L^2(D))$, the square integrable functions $u(\boldsymbol{x},\omega)$ on $D\times\Omega$ such that $u(\cdot,\omega)\in L^2(D)$ for almost every $\omega\in D$, with inner product

$$\langle u,v\rangle_{L^2(\Omega,L^2(D))} := \int_\Omega\int_D u(\boldsymbol{x},\omega)v(\boldsymbol{x},\omega)\,d\boldsymbol{x}\,d\mathbb{P}(\omega).$$

Orthogonal projections and orthonormal bases

Consider a Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and let G be a subspace of H (i.e., $G\subset H$ and is a vector space).

Definition 1.36 (orthogonal complement) The *orthogonal complement* of a closed subspace G of H is

$$G^\perp := \{u\in H : \langle u,v\rangle = 0 \text{ for all } v\in G\}.$$

Note that $G\cap G^\perp = \{0\}$. For a non-zero vector $\boldsymbol{u}\in\mathbb{R}^d$, $G = \operatorname{span}\{\boldsymbol{u}\}$ is a subspace of $\mathbb{R}^d$ and $G^\perp$ is the hyperplane through the origin orthogonal to $\boldsymbol{u}$.

Theorem 1.37 *If G is a closed subspace of a Hilbert space H, every $u\in H$ may be uniquely written $u = p^* + q$ for $p^*\in G$ and $q\in G^\perp$. Further, $\|u-p^*\| = \inf_{p\in G}\|u-p\|$.*