

## Index

- adjoint operator, 159, *see also* Banach adjoint  
 of a matrix transformation, 162  
 of compact operator is compact, 177  
 of shifts, 162  
 of unbounded operator, 265  
 relationship to Banach adjoint, 227  
 self-adjoint, *see* self-adjoint operator  
 spectrum, 168  
 sums and products, 161
- algebra (of continuous functions), 77  
 separates points, 79
- almost everywhere, 94, 308  
 convergence of sequences converging in  $L^p$ , 313
- Apollonius's identity, 108
- approximate eigenvalues, 271
- Arzelà–Ascoli Theorem  
 compact version, 85  
 precompact version, 83
- Axiom of Choice, 301  
 equivalent to Zorn's Lemma, 302
- $B(X)$ , 139
- $B(X, Y)$ , 139  
 complete when  $Y$  is complete, 145  
 norm, 139
- Baire Category Theorem  
 meagre form, 242  
 residual form, 240
- Banach adjoint, 222  
 inherits compactness, 227  
 isomorphisms and isometries, 223  
 relationship to Hilbert adjoint, 227
- Banach Fixed Point Theorem, 63
- Banach Isomorphism Theorem, 251
- Banach limits, 224–226
- Banach space, 54, *see also* completeness  
 complexification, 171
- Banach–Alaoglu Theorem, 293, 328
- Banach–Steinhaus Theorem, *see* Principle of Uniform Boundedness
- basis  
 for a topology, 320  
 Hamel, *see* Hamel basis  
 orthonormal, *see* orthonormal basis  
 Schauder, *see* Schauder basis
- basis constant, 252
- Bernstein polynomials, 72
- Bessel's inequality, 115
- bijjective, 12
- Bolzano–Weierstrass Theorem, 26
- bounded linear operator, 138  
 complexification, 171  
 composition again bounded, 140  
 is continuous, 139  
 kernel is closed, 146  
 norm, 139  
 on a finite-dimensional space, 138  
 range need not be closed, 146  
 spectral radius, 172
- bounded set, 21
- $c_0$ , 6  
 as normed space, 40  
 dual space is  $\ell^1$ , 206, 209  
 is complete, 58, 64  
 is not reflexive, 279  
 is separable, 50, 51
- $c_{00}$ , 6  
 dense in  $\ell^p$ ,  $1 \leq p < \infty$ , 50
- $C([a, b])$ , 5  
 definition of supremum norm, 41

- dual space, 201
- is infinite-dimensional, 9
- is separable, 75
- not complete with  $L^1$  norm, 89
- $C_b(X; \mathbb{K})$ , 41
  - is complete, 59
- $C(X; \mathbb{K})$  is complete, 60
- $C(\overline{\Omega})$  dense in  $L^p(\Omega)$ , 96
- $C^1([a, b])$  is complete, 60
- $C^k([a, b])$ , 61
- Cauchy sequence, 53
- Cauchy–Schwarz inequality, 30
  - in an inner-product space, 103
  - in  $L^2$ , 104
- chain, 14, 301
- characteristic function, 307
- Clarkson’s inequalities, 135
- clin, *see* closed linear span
- closed ball, 36
- closed convex hull, 236
  - of compact set is compact, 239
- Closed Graph Theorem, 255
- closed linear span
  - and linear functionals, 227
  - of an orthonormal set, 124
- closed operator, 267
  - extension of symmetric operator, 268
- closed set, 20
  - definition with sequences, 21
- closed subspace, 48
  - inherits reflexivity, 279
- closure, 24
- compact, 26, 323
  - closed bounded subsets of  $\mathbb{K}$ , 26
  - closed bounded subsets of  $\mathbb{K}^n$ , 28
  - continuous function on a compact set, 28
  - equivalent to sequentially compact in a metric space, 324
  - Hilbert cube, 124
  - Hilbert–Schmidt operator, 176
  - image under a continuous map, 28
  - implies closed and bounded, 27
  - product of compact sets, 27
  - subsets of finite-dimensional spaces, 68
  - unit ball in finite dimensions, 69
- compact operator, 173
  - adjoint is compact, 177
  - and weak convergence, 284
  - composition, 178
  - $K(X, Y)$ , *see*  $K(X, Y)$
  - spectrum, 263
  - spectrum contains 0, 178
  - when also self-adjoint
    - eigenvalues tends to zero, 183
    - eigenvectors form a basis, 185
    - spectrum is closure of point spectrum, 186
- comparable, 14, 301
- completeness, 54
  - and absolutely convergent sequences, 62
  - of  $B(X, Y)$ , 145
  - of  $c_0$ , 58, 64
  - of  $C(X; \mathbb{K})$ , 60
  - of  $C_b(X; \mathbb{K})$ , 59
  - of  $C^1([a, b])$ , 60
  - of  $\mathcal{F}_b(X; \mathbb{K})$ , 58
  - of  $K(X, Y)$ , 174
  - of  $\mathbb{K}^n$ , 54
  - of  $\ell^p$ , 57
  - of  $\ell^\infty$ , 64
  - of  $L^p$ , 96, 311
  - of closed subspaces of complete spaces, 55
  - of finite-dimensional spaces, 67
  - of products, 56
  - of quotient space, 64
  - preserved by equivalent norms, 55
  - preserved under isomorphisms, 55
- completion
  - of a metric space, 98
  - of a normed space, 91
  - unique up to isometric isomorphism, 93
- complex-linear map, 214
- complexification
  - of a Banach space, 171
  - of a bounded linear operator, 171
  - of a compact self-adjoint operator, 189
  - of a Hilbert space, 171
- Condensation of Singularities, 244
  - for a residual set, 247
- conjugate indices, 202
- conjugate-linear map, 10
- continuity
  - equivalent to sequential continuity, 22
  - in metric spaces, 22
  - in terms of a sub-basis, 322
  - in terms of open sets, 22
  - in topological spaces, 320
  - uniform, 29
- continuous function, *see also* continuity
  - on a compact set, 28
- continuous spectrum, 270
- Contraction Mapping Theorem, 63

- convergence
  - and inner products, 106
  - in metric spaces, 21
  - in normed spaces, 43, 64
  - in terms of open sets, 21
  - in topological spaces, 320
- convex function, 37, 51
- convex hull, 235
  - closed, *see* closed convex hull
- convex set, 36
  - and linear functionals, 235
  - closed implies weakly closed, 289
  - closest point in Hilbert space, 126
  - functional separation theorem, 230
- countably additive, 306
- cover, 26
  
- density, 24
  - of  $C(\overline{\Omega})$  in  $L^p(\Omega)$ , 313
- diagonal argument, 84
- dimension, 9
- Dini's Theorem, 86
- direct sum, 129
- discrete metric, 19
- distance functional, 220
  - in Hilbert spaces, 226
- distance to a closed set, 68
- domain of an operator, 265
- Dominated Convergence Theorem, 310
- dual space, 153, 201
  - of  $c_0$ , 206, 209
  - of  $C([a, b])$ , 201
  - of  $\ell^1$ , 206
  - of  $\ell^p$ , 204
  - of  $L^p$ ,  $1 \leq p < \infty$ , 207, 315
    - proof using reflexivity, 281
    - proof using Riesz Representation Theorem, 209
    - proof using uniform convexity, 317
  - of  $\mathbb{R}^n$ , 154
  - separability passed from  $X^*$  to  $X$ , 221
  
- $e^{(j)}$  (basis sequence in  $\ell^p$ ), 9
- eigenfunctions of Sturm–Liouville problem, 190
  - form a basis, 197
- eigenspace, 166
- eigenvalues, 165, 166
  - bounded by operator norm, 166
  - in finite dimensions, 165
  - linearly independent eigenvectors, 166
  - min-max expression, 189
  - multiplicity, 166
  - of compact operator, 258–263
  - of compact self-adjoint operator, 182–183, 186
  - of self-adjoint operators are real, 181
  - of Sturm–Liouville problem
    - are positive, 191
    - are simple, 193
    - real for a symmetric operator, 270
- eigenvectors, 166
  - linearly independent for distinct eigenvalues, 166
  - of compact self-adjoint operator
    - form a basis, 185
- equicontinuous, 83
- equivalent norms, 43
  - define same open sets, 45
  - on finite-dimensional spaces, 66
  - preserve compactness and convergence, 45
  - preserve completeness, 55
- extreme point, 236
- extreme set, 237
  - contains an extreme point, 238
  
- $\mathcal{F}(U, V)$ , 4
- $\mathcal{F}_b(X; \mathbb{K})$  is complete, 58
- factorial, 72
- Fatou's Lemma, 309
- finite-dimensional spaces
  - all are norms equivalent, 66
  - compact subsets, 68
  - have compact unit balls, 69
  - is complete, 67
  - linear operators are bounded, 138
- Fourier series
  - cosine series, 75, 81, 125
  - exponential series, 82, 120
  - non-convergence, 245–247
  - Parseval's identity, 120
  - sine series, 76, 198
- Fourier transform, 172
- Friedrichs extension, 269
- Fubini's Theorem, 163, 310
- functional separation theorem
  - complex version, 232
  - real version, 230
  
- Gram–Schmidt orthonormalisation, 114
- Green's function, 193

- Hahn–Banach Theorem  
 complex version, 216  
 in normed spaces, 219  
 proof in a Hilbert space, 217  
 proof in a separable space, 218  
 real version, 211
- Hamel basis, 8, 16  
 for any vector space, 14  
 non-existence for Banach space, 70, 247
- Hellinger–Toeplitz Theorem, 256
- Helly’s Theorem, 293
- Hilbert adjoint, *see* adjoint operator
- Hilbert cube is compact, 124
- Hilbert space, 107  
 closed subspace is Hilbert, 107  
 complexification, 171  
 is reflexive, 275  
 non-separable example, 123  
 products, 108  
 separable ( $H \equiv \ell^2$ ), 122  
 weak convergence, 282
- Hilbert–Schmidt operator, 176  
 norm bounds operator norm, 178
- Hilbert–Schmidt Theorem, 184
- Hölder’s inequality, 202
- homeomorphism, 31
- hyperplanes, 233  
 and linear functionals, 233
- identity map, 11
- image, *see* range
- induced norm, 104
- infinite-dimensional, 9
- initial segment, 301
- injective, 11  
 iff surjective in finite dimensions, 12
- inner product, 101  
 and convergence, 106  
 induces norm, *see* induced norm  
 on  $\mathbb{K}^n$ , 102  
 on  $\ell^2$ , 102  
 on  $L^2$ , 103
- inner product space, 101  
 complete, *see* Hilbert space
- interior, 23
- inverse, 12  
 linear if map is linear, 12  
 of composition, 13  
 of invertible self-adjoint operator is self-adjoint, 164
- Inverse Mapping Theorem, 251
- invertibility, 147  
 equivalent conditions, 147  
 in finite dimensions, 148  
 is open in  $B(X, Y)$ , 148  
 of commuting operators, 150  
 of products, 149
- isometric isomorphism, 46  
 Banach adjoint, 223  
 preserves reflexivity, 279
- isomorphism, 46  
 Banach adjoint, 223  
 preserves completeness, 55
- $J$  (mapping into second dual), 274
- Jordan–von Neumann Theorem, 108
- $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , 3
- $\mathbb{K}^n$   
 as a normed space, 38  
 inner product, 102  
 is complete, 54  
 standard metric, 18
- $K(X, Y)$  is complete, 174
- kernel  
 of a bounded linear map is closed, 146  
 of a linear map, 11  
 trivial implies injective, 11
- Krein–Milman Theorem, 238
- $L(X, Y)$  and  $L(X)$  are vector spaces, 11
- $\ell^1$ , *see also*  $\ell^p$   
 dual space is  $\ell^\infty$ , 206  
 is not reflexive, 279  
 weak and strong convergence coincide, 286
- $\ell^2$  inner product, 102
- $\ell^p$ , 5  
 as normed space, 40  
 dual space is  $\ell^q$  for  $1 \leq p < \infty$ , 204  
 Hölder’s inequality, 202  
 inclusions, 40  
 is complete, 57  
 is infinite-dimensional, 9  
 is reflexive for  $1 < p < \infty$ , 276  
 is separable for  $1 \leq p < \infty$ , 49  
 Schauder basis, 111  
 weak convergence, 285
- $\ell^\infty$ , *see also*  $\ell^p$   
 dual is not  $\ell^1$ , 222  
 is complete, 64  
 is not reflexive, 279  
 is not separable, 49

- $L^2$ , 95  
 Cauchy–Schwarz inequality, 104  
 inner product, 103  
 $L^p$ , 95, 311  
 $C(\overline{\Omega})$  dense in  $L^p(\Omega)$ , 313  
 dual space is  $L^q$ , 207, 315  
   proof using reflexivity, 281  
   proof using Riesz Representation Theorem, 209  
   proof using uniform convexity, 317  
 Hölder’s inequality, 203  
 inclusions, 209  
 is complete, 96, 311  
 is reflexive for  $1 < p < \infty$ , 277  
 is separable for  $1 \leq p < \infty$ , 313  
 is separable [ $L^p(a, b)$ ], 96  
 is uniformly convex, 135  
 norm by duality, 208  
 $L^\infty$ , 96, 311  
   dual is not  $L^1$ , 222  
 Lax–Milgram Lemma, 158  
 Lebesgue integral, 310  
 Lebesgue measure, 305  
 left shift, *see* shift operators  
 Legendre polynomials, 136  
 linear functionals, 153  
   and convex sets, 235  
   bounded, *see* bounded linear functional  
 linear map, *see* linear operator  
 linear operator, 10  
   bounded, *see* bounded linear operator  
   compact, *see* compact linear operator  
   composition is linear, 11  
   continuous iff bounded, 139  
   has linear inverse, 12  
 linear span, 7  
 linearly independent, 7  
 Lipschitz function, 85  
   approximating continuous functions, 88  
 Lusin’s Theorem, 307  
  
 maximal element, 301  
 maximal linearly independent set, 8  
 Mazur’s Theorem, 289  
 meagre, 242  
 measurable function, 306  
 measurable set, 305  
 measure zero, 94  
 metric, 17  
   derived from a norm, 36  
   discrete, 19  
   on subsets and products, 19  
 metric space, 17  
   bounded set, 21  
   continuity, 22  
   convergence, 21  
   open ball, 20  
   product, 19, 30  
 Milman–Pettis Theorem, 281  
 min-max expression for eigenvalues, 189  
 Minkowski functional, 228  
 Minkowski’s inequality, 208  
   in  $\mathbb{K}^n$ , 39  
   in  $\ell^p$ , 40  
 modulus of continuity, 86  
 Monotone Convergence Theorem, 209, 309  
 multiplicity of eigenvalues, 166  
  
 $\binom{n}{k}$ , 72  
 Neumann series, 151  
 norm, 35  
   all equivalent on a finite-dimensional space, 66  
   equivalent, *see* equivalent norms  
   gives rise to a metric, 36  
   induced by inner product, *see* induced norm  
    $L^p$ , 42  
   of bounded linear operator, 139  
   on  $B(X, Y)$ , 139  
     alternative definitions, 150  
   on finite-dimensional space, 39  
   on  $X^*$  (dual space), 153  
   supremum, 41  
 normed space, 35  
   complete, *see* Banach space  
   convergence, 42  
 nowhere dense, 241  
 null sequences, 6  
 numerical range, 180, 188, 226  
  
 one-to-one, *see* injective  
 onto, *see* surjective  
 open ball  
   in metric spaces, 20  
   in normed spaces, 36  
 Open Mapping Theorem, 249  
 open set, 20  
   is union of open balls, 31  
 ordinary differential equation  
   with continuous right-hand side, 88  
   with Lipschitz right-hand side, 64  
 orthogonal, 112

- orthogonal complement, 129
- orthogonal projection, 129
- orthogonal series
  - convergence, 116
- orthonormal basis, 117
  - criteria, 118
  - for any Hilbert space, 125
  - in finite-dimensional spaces, 115
- orthonormal set, 112
- via Gram–Schmidt process, 114
- outer measure, 305
  
- parallelogram law, 105
  - characterises inner products, 108
- Parseval’s identity, 116, 118, 124
  - for Fourier series, 120
- partial order, 13, 301
- point spectrum, 166, 270
- pointwise convergence, 51
  - and weak convergence, 288
- polarisation identity, 106
- polynomials, 5
  - dense in  $L^p(a, b)$ , 96
  - orthogonal in  $L^2(-1, 1)$ , 132
  - spectral mapping theorem, 170
- precompact, 83
- preimage, 22
- Principle of Uniform Boundedness, 242
- product
  - metric, 19
  - of compact sets is compact, 27
  - of compact topological spaces is compact, 327
  - of complete spaces is complete, 56
  - of Hilbert spaces, 108
  - of invertible maps, 149
  - of metric spaces, 30
  - of vector spaces, 4
  - topology, 326
- Pythagoras Theorem, 113
  
- $\mathbb{Q} + i\mathbb{Q}$ , 25
- quotient space, 15
  - in Hilbert spaces, 109, 136
  - is a vector space, 15
  - is complete, 64
  
- Radon–Nikodym Theorem, 315
- range, 11
  - finite-dimensional implies compact, 173
  - of a bounded linear map need not be closed, 146
- real-linear map, 214
- reflexivity, 275
  - and weak sequential compactness, 294, 297
  - $c_0$  not reflexive, 279
  - $C([-1, 1])$  not reflexive, 277
  - inherited by closed subspaces, 279
  - $\ell^1$  and  $\ell^\infty$  not reflexive, 279
  - of Hilbert spaces, 275
  - of  $L^p$  for  $1 < p < \infty$ , 277
  - of  $\ell^p$  for  $1 < p < \infty$ , 276
  - preserved under isometric isomorphisms, 279
  - $X$  reflexive iff  $X^*$  reflexive, 278
- residual, 240
- residual spectrum, 270
  - empty for self-adjoint operators, 271
- resolvent set
  - for bounded operators, 165
  - for maps between Banach spaces, 251
  - for unbounded operators, 270
- Riesz map, 155
  - is isometric isomorphism when  $H$  is real, 202
- Riesz Representation Theorem, 155
- Riesz’s Lemma, 68
- right shift, *see* shift operators
  
- Schauder basis, 110
  - characterisation of a basis, 256
  - for  $\ell^p$ , 111
  - in Banach spaces, 252
- Schur’s Theorem, 286
- second dual, 273
  - canonical mapping  $J$ , 274
- self-adjoint operator, 161
  - closed in  $B(H)$ , 164
  - for unbounded operators, 266
  - has real eigenvalues, 181
  - inverse is self-adjoint if bounded, 164
  - orthogonality of eigenfunctions, 181
  - residual spectrum is empty, 271
  - when also compact
    - eigenvalues tend to zero, 183
    - eigenvectors form a basis, 185
    - spectrum is closure of point spectrum, 186
- seminorm, 211, 217
- separability, 25
  - equivalent characterisations, 48

- of  $c_0$ , 50, 51
- of  $C([a, b])$ , 75
- of  $\ell^p$ , 49
- of  $L^p$ ,  $1 \leq p < \infty$ , 313
- of  $L^p(a, b)$ , 96
- of products, 31
- passes from  $X^*$  to  $X$ , 221
- passes to subsets, 25
- sequence
  - Cauchy, *see* Cauchy sequence
  - in Banach space
    - absolute convergence implies convergence, 61
- sequential continuity, 31
  - equivalent to continuity, 22
- sequentially compact, 26, 323
  - equivalent to compact in a metric space, 324
- shift operators, 141
  - adjoints, 162
  - eigenvalues and spectrum, 167–169
  - not invertible, 150
- $\sigma$ -algebra, 306
- simple function, 306
- Spectral Mapping Theorem, 170, 183
- spectral radius, 172
- spectrum
  - in finite dimensions, 166
  - of adjoint operators, 168
  - of bounded operators, 166
    - compact subset of  $\mathbb{C}$ , 167
  - of compact operators, 263
    - contains 0, 178
  - of maps between Banach spaces, 251
  - of unbounded operators, 270–272
- \*\* (map into second dual), 273
- Stone–Weierstrass Theorem
  - complex version, 81
  - real version, 79
- strictly convex Banach space, 135, 226
- strong convergence, 282
- Sturm–Liouville problem, 190
  - eigenfunctions form a basis, 197
  - eigenvalues are positive, 191
  - eigenvalues are simple, 193
  - integral formulation, 195
  - orthogonality of eigenfunctions, 191
- sub-basis (for a topology), 321
- subalgebra, *see* algebra
- subcover, 26
- sublinear functional, 211
- subsequence, 26
- subspace, 6
  - finite-dimensional subspace is closed, 70
  - of normed space need not be closed, 50
- support functional, 219
  - in Hilbert spaces, 226
- supremum norm, 41
- surjective, 12
  - iff injective in finite dimensions, 12
- symmetric operator, 161
  - eigenvalues are real, 270
- Tonelli’s Theorem, 310
- topological space, 320
- topology, 319
  - basis, 320
  - on product space, 326
  - sub-basis, 321
  - weak-\*, 322
- totally bounded, 87
- totally ordered, 301
- triangle inequality, 17, 35
- Tychonoff’s Theorem, 327
- unbounded operator, 265
  - adjoint, 265
  - resolvent and spectrum, 269
  - self-adjoint, 266
    - spectrum is real and closed, 272
- uniform continuity, 29
- uniform convergence, 58
- uniformly convex Banach space, 109
  - closest point in a convex set, 136
  - $L^p$ , 135
  - weak plus norm convergence implies strong, 296
- upper bound, 14, 301
- vector space, 3
  - always has a Hamel basis, 14, 16
  - dimension, 9
  - product, 4
  - subspace, 6
- weak convergence, 282–283
  - and pointwise convergence, 288
  - becomes strong under action of a compact operator, 284
  - in a uniformly convex Banach space
    - becomes strong with convergence of norms, 296
  - in Hilbert spaces, 282

- becomes strong with convergence of norms, 284
  - in  $\ell^1$  is strong, 286
  - in  $\ell^p$ , 285
- weak-\* compactness in separable spaces, 293
- weak-\* convergence, 290
- weak-\* topology, 322
  - and weak-\* convergence, 322
  - metrisable on  $\mathbb{B}_{X^*}$ , 329
- weakly closed, 289
  - closed convex sets are weakly closed, 289
- weakly sequentially compact, 294
  - if and only if reflexive, 297
- unit ball in reflexive space, 294
- Weierstrass Approximation Theorem, 71
- well-ordered set, 302
- Wronskian, 192
- $X^* = B(X; \mathbb{K})$  (dual space), 153
- $x^{**}$  (map into second dual), 273
- Young's inequality, 202, 208
- Zorn's Lemma, 14, 125, 211, 238, 326
  - equivalent to Axiom of Choice, 302