Cambridge University Press
978-0-521-89964-2 — An Introduction to Functional Analysis
James C. Robinson
Frontmatter
More Information

# An Introduction to Functional Analysis

This accessible text covers key results in functional analysis that are essential for further study in the calculus of variations, analysis, dynamical systems, and the theory of partial differential equations. The treatment of Hilbert spaces covers the topics required to prove the Hilbert–Schmidt Theorem, including orthonormal bases, the Riesz Representation Theorem, and the basics of spectral theory. The material on Banach spaces and their duals includes the Hahn–Banach Theorem, the Krein–Milman Theorem, and results based on the Baire Category Theorem, before culminating in a proof of sequential weak compactness in reflexive spaces. Arguments are presented in detail, and more than 200 fully-worked exercises are included to provide practice applying techniques and ideas beyond the major theorems. Familiarity with the basic theory of vector spaces and point-set topology is assumed, but knowledge of measure theory is not required, making this book ideal for upper undergraduate-level and beginning graduate-level courses.

JAMES ROBINSON is a professor in the Mathematics Institute at the University of Warwick. He has been the recipient of a Royal Society University Research Fellowship and an EPSRC Leadership Fellowship. He has written six books in addition to his many publications in infinite-dimensional dynamical systems, dimension theory, and partial differential equations.

# An Introduction to Functional Analysis

JAMES C. ROBINSON
*University of Warwick*

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of
education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9780521899642
DOI: 10.1017/9781139030267



First published 2020

Printed in the United Kingdom by TJ International Ltd. Padstow Cornwall

*A catalogue record for this publication is available from the British Library.*

ISBN 978-0-521-89964-2 Hardback
ISBN 978-0-521-72839-3 Paperback

To Mum & Dad

# Contents

Cambridge University Press
978-0-521-89964-2 — An Introduction to Functional Analysis
James C. Robinson
Frontmatter
More Information

# Preface

This book is intended to cover the core functional analysis syllabus and, in particular, presents many of the results that are needed in partial differential equations, the calculus of variations, or dynamical systems. The material is developed far enough that the next step would be application to one of these areas or further pursuit of 'functional analysis' itself at a significantly more advanced level.

The content is based on the two functional analysis modules taught at the University of Warwick to our third-year undergraduates. As such, it should be straightforward to use this book (with some judicious pruning) as the basis of a two-term course, with Part III (Hilbert spaces) taught in the first term and Part IV (Banach spaces) in the second term. Part II contains foundational material (a general theory of normed spaces and a collection of example spaces) that is needed for both Parts III and IV; some of this material could find a home in either term, according to taste. A one-term standalone module on Banach spaces could be based on Part II; Chapters 11, 14, and 15 from Part III; and Part IV.

Familiarity is assumed with the theory of finite-dimensional vector spaces and basic point-set topology (metric spaces, open and closed sets, compactness, and completeness), which is revised, at a fairly brisk pace and with some proofs omitted, in the first two chapters. No knowledge of measure theory or Lebesgue integration is required: the Lebesgue spaces are introduced as completions of the space of continuous functions in Chapter 7, with the standard construction of the Lebesgue integral outlined in Appendix B. The canonical examples of non-Hilbert Banach spaces used in Part IV are the sequence spaces $\ell^p$ rather than the Lebesgue spaces $L^p$; I hope that this will make the book accessible to a wider audience. In the same spirit I have tried to spell out all the arguments in detail; there are no[1] four-line proofs that when written with all the details expand to fill the same number of pages.

For the most part the approach adopted here is to cover the simpler case of Hilbert spaces in Part III before turning to Banach spaces, for which the theory becomes more abstract, in Part IV. There is an argument that it is more efficient to prove results in Banach spaces before specialising to Hilbert spaces, but my suspicion is that this is a product of familiarity and experience: in the same way one might argue that it is more economical to teach analysis in metric spaces before specialising to the particular case of real sequences and real-valued functions. That said, some basic concepts and results are not significantly simpler in Hilbert spaces, so portions of Parts II and III deal with Banach rather than Hilbert spaces.

By way of a very brief overview of the contents of the book, it is perhaps useful to describe the end points of Parts III and IV. Part III works towards the Hilbert–Schmidt Theorem that decomposes a self-adjoint compact operator on a Hilbert space in terms of its eigenvalues and eigenfunctions, and then applies this to the example of the Sturm–Liouville eigenvalue problem. It therefore covers orthonormal bases, orthogonal projections, the Riesz Representation Theorem, and the basics of spectral theory. Part IV culminates with the result that the closed unit ball in a reflexive Banach space is weakly sequentially compact. So this part covers dual spaces in more detail, the Hahn–Banach Theorem and applications to convex sets, results for linear operators based on the Baire Category Theorem, reflexivity, and weak and weak-$*$ convergence.

Almost every chapter ends with a collection of exercises, and full solutions to these are given at the end of the book.

There are three appendices. The first shows the equivalence of Zorn's Lemma and the Axiom of Choice; the second provides a quick overview of the construction of the Lebesgue integral and proves properties of the Lebesgue spaces that rely on measure-theoretic techniques; and the third proves the Banach–Alaoglu Theorem on weak-$*$ compactness of the closed unit ball in an arbitrary Banach space, a topological result that lies outside the scope of the main part of the book.

I am indebted to those at Warwick who taught the Functional Analysis courses before me, both in the selection of the material and the general approach. Although I have adapted both over the years, the skeleton of this book was provided by Robert MacKay and Keith Ball, to whom I am very

[1] Actually, there is one. An abridged version of the proof that $(L^p)^* \equiv L^q$ appears in Chapter 18 and takes about half a page. The detailed proof, which requires some non-trivial measure theory, takes up two pages Appendix B.

grateful. Those who have subsequently taught the same material, Richard Sharp and Vassili Gelfreich, have also been extremely helpful.

Writing a textbook encourages a magpie approach to results, proofs, and examples. I have been extremely fortunate that there are already a large number of texts on functional analysis, and I have tried to take advantage of the many insights and the imaginative problems that they contain. Just as there are standard results and standard proofs, there are many standard exercises, but I have credited those that I have adopted that seemed particularly imaginative or unusual. In addition, there is a long list of references at the back of the book, and each of these has contributed something to this text. I would particularly like to acknowledge the book by Rynne and Youngson (2008) and the older texts by Kreyszig (1978) and Pryce (1973) as consistent sources of inspiration. The books by Giles (2000) and Lax (2002) contain many interesting examples and exercises.

I have not tried to trace the history of the many now 'classical' results that occur throughout the book. For those who are interested in this aspect of the subject, Giles (2000) has an appendix that gives a nice overview of the historical background, and historical comments are woven throughout the text by Lax (2002). Banach's 1932 monograph contains a significant proportion of the results in Part IV.

Many staff at Cambridge University Press have been involved with this project over the years: Clare Dennison, Sam Harrison, Amy He, Kaitlin Leach, Peter Thompson, and David Tranah. Given such a long list of names, it goes without saying that I would like to thank them all for their patience and support (and apologise to anybody I have missed). I would particularly like to thank Kaitlin for ultimately holding me to a deadline that meant I finally finished the book.

Lastly, I am extremely grateful to Wojciech Ożański, who read a draft version of this book and provided me with many corrections, suggestions, and insightful comments.

Cambridge University Press
978-0-521-89964-2 — An Introduction to Functional Analysis
James C. Robinson
Frontmatter
More Information