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PART I: GAMES AND SCALES

GAMES AND SCALES
INTRODUCTION TO PART I

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The construction and use of *Suslin representations* for sets of reals lies at the heart of descriptive set theory. Indeed, virtually every paper in descriptive set theory in the Cabal Seminar volumes deals with such representations in one way or another. Most of the papers in the section to follow focus on the construction of optimally definable Suslin representations via game-theoretic methods. In this introduction, we shall attempt to put those papers in a broader historical and mathematical context. We shall also give a short synopsis of the papers themselves, and describe some of the work done later to which they are related.

§1. Some definitions and history. A **tree** on a set X is a subset of $X^{<\omega}$ closed under initial segments. If T is a tree on $X \times Y$, then we regard T as a set of pairs (s, t) of sequences with $\mathrm{dom}(s) = \mathrm{dom}(t)$. If T is a tree, we use $[T]$ for the set of infinite branches of T, and if T is on $X \times Y$, we write

$$\mathrm{p}[T] = \{x \in {}^{\omega}X \,:\, \exists y \in {}^{\omega}Y \forall n < \omega((x \restriction n, y \restriction n) \in T)\}.$$

We call $\mathrm{p}[T]$ the **projection** of T, and say that T is a **Suslin representation** of $\mathrm{p}[T]$, or that $\mathrm{p}[T]$ is Y**-Suslin** via T. For $s \in X^{<\omega}$, let $T_s = \{u : (s, u) \in T\}$, and put $T_x = \bigcup_n T_{x \restriction n}$. Then $x \in \mathrm{p}[T]$ iff $[T_x] \neq \varnothing$ iff T_x is illfounded.

Any set $A \subseteq {}^{\omega}X$ is trivially A-Suslin. For the most part, useful Suslin representations come from trees on some $X \times Y$ such that Y is wellordered. Assuming (as we do) the Axiom of Choice (AC), this is no restriction on Y, but we can parlay it into an important and useful restriction by requiring in addition that T be *definable* in some way or other. A variant of this approach is to require that T belong to a model of AD. If T is definable, and X and Y are definably wellordered, and $\mathrm{p}[T]$ is nonempty, then the leftmost branch (x, f) of T gives us a definable element x of $\mathrm{p}[T]$. (Here "leftmost" can be determined by the lexicographic order on $X \times Y$.)

The simplest nontrivial X to consider are the countable ones. This is by far the most well-studied case in the Cabal volumes. In this case, one may regard $\mathrm{p}[T]$ as a subset of the Baire space ${}^{\omega}\omega$, that is, as a set of "logician's reals".

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Thus if A is a nonempty set of reals, κ is an ordinal, and A is κ-Suslin via a definable tree, then A has a definable element.

Suslin representations were first discovered in 1917 by Suslin [Sus17], who isolated the class of ω-Suslin sets of reals, showed that it properly includes the Borel sets, and showed that sets in this class have various regularity properties. (For example, they are all Baire and Lebesgue measurable, and have the perfect set property.) Suslin also found a beautiful characterization of the Borel sets of reals as those which are both ω-Suslin and have ω-Suslin complements. (The ω-Suslin sets of reals are precisely the $\undertilde{\mathbf{\Sigma}}^1_1$ sets of reals, almost by definition.)

Definable Suslin representations yield definable elements, and in the "boldface" setting of classical descriptive set theory, this comes out as a uniformization result. Here we say that a function f **uniformizes** a relation R iff $\mathrm{dom}(f) = \{x : \exists y R(x, y)\}$, and $\forall x \in \mathrm{dom}(f) R(x, f(x))$. If R is a $\undertilde{\mathbf{\Sigma}}^1_1$ relation, say $R = \mathrm{p}[T]$ where T is a tree on $(\omega \times \omega) \times \omega$, then we can use leftmost branches to uniformize R: let $f(x) = y$, where (y, h) is the leftmost branch of T_x. One can calculate that for any open set U, $f^{-1}(U)$ is in the σ-algebra generated by the $\undertilde{\mathbf{\Sigma}}^1_1$ sets, and is therefore Lebesgue and Baire measurable. This classical uniformization result was proved by Jankov and von Neumann around 1940 [vN49]. The "lightface", effective refinement of a uniformization theorem is a *basis theorem*, where we say a pointclass Λ is a basis for a pointclass Γ just in case every nonempty set of reals in Γ has a member which is in Λ. Kleene [Kle55] proved the lightface version of the Jankov-von Neumann result. He observed that if $A \subseteq \omega^\omega$ is lightface Σ^1_1, then $A = \mathrm{p}[T]$ for some recursive tree T, and that the leftmost branch of T is recursive in the set W of all Gödel numbers of wellfounded trees on ω. Thus $\{x : x \leq_T W\}$ is a basis for Σ^1_1.

In 1935–38, toward the end of the classical period, Novikoff and Kondô constructed definable, ω_1-Suslin representations for arbitrary $\undertilde{\mathbf{\Sigma}}^1_2$ sets, and used them to show every $\undertilde{\mathbf{\Sigma}}^1_2$ relation has a $\undertilde{\mathbf{\Sigma}}^1_2$ uniformization. (See [LN35, Kon38].) The effective refinement of this landmark theorem is due to Addison, who showed that the ω_1-Suslin reprentations of nonempty lightface Σ^1_2 sets constructed by Novikoff and Kondô yield, via leftmost branches, lightface Δ^1_2 elements for such sets.

Logicians often meet Suslin representations through the Shoenfield Absoluteness theorem. Shoenfield [Sch61] showed that a certain tree T on $\omega \times \omega_1$ which comes from the Novikoff-Kondô construction is in $\mathbf{L}$. Because wellfoundedness is absolute to transitive models of ZF, he was able to conclude that the leftmost branch of T is in $\mathbf{L}$, and thus, that every nonempty Σ^1_2 set of reals has an element in $\mathbf{L}$. From this it follows easily that $\mathbf{L}$ is Σ^1_2 correct. This method of using definable Suslin representations to obtain correctness and absoluteness results for models of set theory is very important.

In addition to definability, there is a second very useful property a Suslin representation might have. We call a tree T on $X \times Y$ **homogeneous** just in case there is a family $\langle \mu_s : s \in X^{<\omega} \rangle$ such that

(1) for all s, μ_s is a countably complete 2-valued measure (i.e. ultrafilter) on $\{u : (s,u) \in T\}$,
(2) if $s \subseteq t$, and $\mu_s(A) = 1$, then $\mu_t(\{u : u \restriction \mathrm{dom}(s) \in A\}) = 1$, and
(3) for any $x \in \mathrm{p}[T]$ and any $\langle A_i : i < \omega \rangle$ such that $\mu_{x \restriction i}(A_i) = 1$ for all i, there is a $f \in Y^\omega$ such that $f \restriction i \in A_i$ for all i.

We say T is **κ-homogeneous** if the measures μ_s can be taken to be κ-additive. If T is κ-homogeneous, then we also call $\mathrm{p}[T]$ a κ-homogeneously Suslin set. We write Hom_κ for the pointclass of κ-homogeneous sets, and Hom_∞ for the pointclass $\bigcap_\kappa \mathrm{Hom}_\kappa$.

The concept of homogeneity is implicit in Martin's 1968 proof [Mar70A] of Π^1_1 determinacy, and was first explicitly isolated by Martin and Kechris. Martin showed that if κ is a measurable cardinal, then every Π^1_1 set of reals is κ homogeneous, via a Shoenfield tree on $\omega \times \kappa$. He also showed that every homogeneously Suslin set of reals is determined. Martin's proof became the template for all later proofs of definable determinacy from large cardinal hypotheses. Indeed, the standard characterization of descriptive set theory, as the study of the good behavior of definable sets of reals, would perhaps be more accurate if one replaced "definable" by "∞-homogeneously Suslin".

There are two natural weakenings of homogeneity. First, a tree T on $X \times (\omega \times Y)$ is **κ-weakly homogeneous** just in case it is κ- homogeneous when viewed as a tree on $(X \times \omega) \times Y$. Thus the weakly homogeneous subsets of ${}^\omega X$ are just the existential real quantifications of a homogeneous subsets of ${}^\omega X \times {}^\omega\omega$, and Martin's [Mar70A] shows in effect that whenever κ is measurable, all $\utilde{\Sigma}^1_2$ sets of reals are κ-weakly homogeneous. Second, a pair of trees S and T, on $X \times Y$ and $X \times Z$ respectively, are **κ-absolute complements** iff

$$\mathbf{V}[G] \models \mathrm{p}[S] = {}^\omega X \setminus \mathrm{p}[T]$$

whenever G is $\mathbf{V}$-generic for a poset of cardinality $< \kappa$. The fundamental Martin-Solovay construction, also from 1968 (see [MS69]), shows that every κ-weakly homogeneous tree has a κ-absolute complement. The projection of a κ-absolutely complemented tree is said to be **κ-universally Baire**. This concept was first explicitly isolated and studied by Feng, Magidor, and Woodin in [FMW92]. Any universally Baire set has the Baire property and is Lebesgue measurable, but one cannot show in ZFC alone that such sets must be determined. (See [FMW92].) On the other hand, if there are arbitrarily large Woodin cardinals, then for any set of reals A, A is κ-homogeneous for all κ iff A is κ-weakly homogeneous for all κ iff A is κ-universally Baire for all κ. (This is work of Martin, Solovay, Steel, and Woodin; see [Lar04, Theorem 3.3.13] for one exposition, and [SteA] for another.)

Although our discussion of homogeneity has focussed on its use in situations where the Axiom of Choice and the existence of large cardinals is assumed, the concept is also quite important in contexts in which full AD is assumed. AD gives us not just measures, but homogeneity measures; indeed, assuming AD, a set of reals is homogeneously Suslin iff both it and its complement are Θ-Suslin. (This result of Martin from the 80's can be found in [MS89].) The analysis of homogeneity measures is a central theme in the work of Kunen, Martin, and Jackson [Sol78A, Jac88, Jac99] which located the projective ordinals among the alephs. The reader should see Jackson's surveys [Jac07A] and [Jac07B] for more on homogeneity and the projective ordinals in the AD context.

§2. **Construction methods.** One could group the methods for producing useful Suslin representations as follows:

(1) the Martin-Solovay construction,
(2) trees to produce an elementary submodel, and
(3) scale constructions using comparison games.

We discuss these methods briefly:

2.1. The Martin-Solovay construction. The Martin-Solovay construction makes use of homogeneity. If T on $X \times Y$ is is κ- weakly homogeneous via the system of measures $\vec{\mu}$, and $|X| < \kappa$, then the construction produces a tree $\mathrm{ms}(T, \vec{\mu})$ which is a κ-absolute complement for T. The construction of $\mathrm{ms}(T, \vec{\mu})$ is effective, and its basic properties can be proved to hold in $\mathsf{ZF} + \mathsf{DC}$. Martin and Solovay [MS69] applied it with T the Shoenfield tree for Σ^1_2 and $\vec{\mu}$ its weak homogeneity measures implicit in Martin's [Mar70A]. They showed thereby that if κ is measurable, then for any Σ^1_3 formula φ, there is a tree U such that $\mathrm{p}[U] = \{x \in \omega^\omega : \varphi(x)\}$ is true in every generic extension of $\mathbf{V}$ by a poset of size $< \kappa$.

The Martin-Solovay tree $\mathrm{ms}(T, \vec{\mu})$ is definable from T and $\vec{\mu}$. Now suppose T be on $\omega \times Y$. There is a simple variant of $\mathrm{ms}(T, \vec{\mu})$ which is definable from T and the restrictions of the measures in $\vec{\mu}$ to $\bigcup\{\mathbf{L}[T, x] : x \in {}^\omega\omega\}$. Let us call this variant $\mathrm{ms}^*(T, \vec{\mu})$. If T is the Shoenfield tree, so that $T \in \mathbf{L}$, then one can define these restricted weak homogeneity measures, and hence $\mathrm{ms}^*(T, \vec{\mu})$ itself, from the sharp function on the reals. Martin and Solovay showed this way that Δ^1_4 is a basis for Π^1_2, and Mansfield later improved their result by showing the class of Π^1_3 singletons is a basis for Π^1_2. (See [Man71].) These results are not optimal, however. We do not know whether one can get the optimal basis and uniformization results in the projective hierarchy using the Martin-Solovay construction.

Under appropriate large cardinal hypotheses, the Martin-Solovay tree is itself homogeneous. (See [MS07] for a precise statement.) Thus under the appropriate large cardinal hypotheses, one can show via the Martin-Solovay construction that the pointclass Hom_∞ is closed under complements and real quantification.

2.2. The tree to produce an elementary submodel. If a set A of reals admits a definition with certain condensation and generic absoluteness properties, then A is universally Baire. More precisely, let κ be a cardinal, and $\varphi(v_0, v_1)$ a formula in the language of set theory, and t any set. Let $\tau > \kappa$, $X \prec \mathbf{V}_\tau$ be countable, and let M be the transitive collapse of X, with $\bar{\kappa}$ and $\bar{t}$ the images of κ and t under collapse. We say X is **generically** $\langle\varphi, A\rangle$**-correct** iff whenever g is M-generic for a poset of size $< \bar{\kappa}$ in M, then for all reals $y \in M[g]$,

$$y \in A \Leftrightarrow M[g] \models \varphi[y, \bar{t}].$$

If the set of generically $\langle\varphi, A\rangle$ correct X is club in $\wp_{\omega_1}(\mathbf{V}_\tau)$, then A admits a κ-absolutely complemented Suslin representation T. The construction of T is relatively straightforward: if $(y, f) \in [T]$, then f will have built an X in our club of generically correct hulls, together with a proof that $M[g] \models \varphi[y, \bar{t}]$, for some g generic over the collapse M of X. (We are not certain as to the origin of this construction. Woodin made early use of it. See [FMW92] or [SteA].)

One can use either stationary tower forcing (cf. the *Tree Production Lemma*, [Lar04] or [SteA]) or iterations to make reals generic [Ste07B, § 7] to obtain, for various interesting $\langle\varphi, A\rangle$, a club of generically $\langle\varphi, A\rangle$-correct X.

If one replaces $\mathbf{V}_\tau$ by an appropriate direct limit of mice, then the tree to produce an elementary submodel becomes definable, at a level corresponding to the definability of the iteration strategies for the mice in question. See the concluding paragraphs of [Ste95A], and [Ste07B, § 8]. One can use this to get optimal basis and uniformization results for various pointclasses, for example $(\Sigma^2_1)^{\mathbf{L}(\mathbb{R})}$. It is difficult to obtain the optimal basis and uniformization results for Π^1_3 by these methods, but, building on work of Hugh Woodin, Itay Neeman has succeeded in doing so. (This work is unpublished.)

2.3. Propagation of scales using comparison games. The simplest method for obtaining optimally definable Suslin representations makes direct use of the determinacy of certain infinite games. It was discovered in 1971 by Moschovakis, who used it to extend the Novikoff-Kondô-Addison theorems to the higher levels of the projective hierarchy. (The original paper is [Mos71A]; see also [KM78B] and [Mos80, Chapter 6].) As part of this work, Moschovakis introduced the basic notion of a *scale*, which we now describe.

Let T be a tree on $\omega \times \lambda$, and $A = \mathrm{p}[T]$. One can get a "small" subtree of T which still projects to A by considering only ordinals $< \lambda$ which appear in some leftmost branch. The *scale of* T does this, then records the resulting subtree as a sequence of **norms**, i.e. ordinal-valued functions, on A. More precisely, for $x \in A$ and $n < \omega$, put

$$\varphi_n(x) = |\langle l_x(0), \ldots, l_x(n)\rangle|_{\mathrm{lex}},$$

where for $u \in \lambda^{n+1}$, $|u|_{\mathrm{lex}}$ is the ordinal rank of u in the lexicographic order on λ^n. Then

$$\vec{\varphi} = \langle \varphi_n : n < \omega \rangle$$

is the scale of T. It has the properties:

(a) Suppose that $x_i \in A$ for all $i < \omega$, and $x_i \to x$ as $i \to \infty$, and for all n, $\varphi_n(x_i)$ is eventually constant as $n \to \infty$, then
  (i) (limit property) $x \in A$, and
  (ii) (lower semi-continuity) for all n, $\varphi_n(x) \leq$ the eventual value of $\varphi_n(x_i)$ as $i \to \infty$.

(b) (refinement property) if $x, y \in A$ and $\varphi_n(x) < \varphi_n(y)$, then $\varphi_m(x) < \varphi_m(y)$ for all $m > n$.

A sequence of norms on A with property (a) is called a **scale on** A. Any scale on A can be easily transformed into a scale on A with the refinement property. If $\vec{\varphi}$ is a scale on A, then we define the tree of $\vec{\varphi}$ to be

$$T_{\vec{\varphi}} = \{(\langle x(0), ..., x(n-1)\rangle, \langle \varphi_0(x), ..., \varphi_{n-1}(x)\rangle) \ : \ n < \omega \text{ and } x \in A\}.$$

It is not hard to see that $\mathrm{p}[T_{\vec{\varphi}}] = A$. If $\vec{\varphi}$ has the refinement property, and $\vec{\psi}$ is the scale of $T_{\vec{\varphi}}$, then $\vec{\psi}$ is equivalent to $\vec{\varphi}$, in the sense that for all n, x and y, $\psi_n(x) \leq \psi_n(y)$ iff $\varphi_n(x) \leq \varphi_n(y)$. The reader should see [KM78B, 6B] and [Jac07B, § 2] for more on the relationship between scales and Suslin representations.

There are least two benefits to considering the scale of a tree: first, it becomes easier to state and prove optimal definability results, and second, the construction of Suslin representations using comparison games becomes clearer. Concerning definability, we have

Definition 2.1. Let Γ be a pointclass, and $\vec{\varphi}$ a scale on A, where $A \in \Gamma$; then we call $\vec{\varphi}$ a **Γ-scale on** A just in case the relations

$$R(n, x, y) \Leftrightarrow x \in A \wedge (y \notin A \vee \varphi_n(x) \leq \varphi_n(y)),$$

and

$$S(n, x, y) \Leftrightarrow x \in A \wedge (y \notin A \vee \varphi_n(x) < \varphi(y))$$

are each in Γ. We say Γ **has the scale property** just in case every set in Γ admits a Γ-scale, and write Scale(Γ) in this case.

Moschovakis showed that if Γ is a pointclass which is closed under universal real quantification, has other mild closure properties, and has the scale property, then every Γ relation has a Γ uniformization, and the Γ singletons are a basis for Γ. [KM78B, 3A-1]. He also showed that assuming $\undertilde{\mathbf{\Delta}}^1_{2n}$ determinacy, both Π^1_{2n+1} and Σ^1_{2n+2} have the scale property [KM78B, 3B, 3C]. From this, one gets the natural generalization of Novikoff-Kondô-Addison to the higher levels of the projective hierachy.

Moschovakis' construction of scales goes by *propagating* them from a set A to a set B obtained from A via certain logical operations. One starts with the fact that Σ^0_1 has the scale property, and uses these propagation theorems to obtain definable scales on more complicated sets. The propagation works at the level of the individual norms in the scales.

For example, if φ is a norm of A, where $A \subseteq X \times {}^{\omega}Y$, and

$$B(y) \Leftrightarrow \exists x A(x, y),$$

then we obtain the "inf" norm on B by setting

$$\psi(y) = \inf\{\varphi(x, y) \,:\, A(x, y)\}.$$

If either X is an ordinal, or $X = {}^{\omega}\omega$, then inf norms can be used to transform a scale on A into a scale on B. (See [KM78B, 3B-2].) This transformation has a simple meaning in terms of the tree of the scale; if $X = {}^{\omega}\omega$, it corresponds to regarding a tree on $(Y \times \omega) \times \kappa$ as a tree on $Y \times (\omega \times \kappa)$.

Definable scales do not propagate under negation or universal quantification over ordinals. (Otherwise, it would be possible to assign to each countable ordinal α a scale on the set of wellorders of ω of order type α, in a definable way. This would then yield a definable function picking a codes for the countable ordinals.) Moschovakis' main advance in [Mos71A] was to show that universal quantification over the reals propagates definable scales. Here it is definitely important to work with scales, rather than their associated trees. As before, the propagation takes place at the level of individual norms. Let φ be a norm on A, where $A \subseteq \mathbb{R} \times Y$, and let

$$B(y) \Leftrightarrow \forall x A(x, y).$$

To each $y \in B$, we associate $f_y : \mathbb{R} \to \mathrm{OR}$, where

$$f_y(x) = \varphi(x, y).$$

Our norm on B records an ordinal measure of the growth rate of f_y. Namely, given $f, g : \mathbb{R} \to \mathrm{OR}$, we let $G(f, g)$ be the game on ω: I plays out x_0, II plays out x_1, the players alternating moves as usual. Player II wins iff $f(x_0) \leq g(x_1)$. (Thus a winning strategy for II witnesses that g grows at least as fast as f, in an effective way.) Now put

$$f \leq^* g \Leftrightarrow \text{II has a winning strategy in } G(f, g).$$

Granted full AD, one can show $\leq^*$ is a prewellorder of all the ordinal-valued functions on $\mathbb{R}$, and granted only determinacy for sets simply definable from φ, one can show that $\leq^*$ prewellorders the f_y for $y \in B$. Our norm on B is then given by

$$\psi(y) = \text{ ordinal rank of } f_y \text{ in } \leq^* \restriction \{f_z \,:\, B(z)\}.$$

(See [KM78B, 2C-1].) The norm ψ is generally called the "fake sup" norm obtained from φ; the ordinal $\psi(y)$ measures how difficult it is to verify $A(x, y)$ at arbitrary x.

The fake-sup construction was first used in [AM68], to propagate the prewellordering property, which involves only one norm. Granted enough determinacy, the construction can be used to transform a scale on A into a scale on B, where $B(y) \Leftrightarrow \forall x A(x, y)$. The key additional idea is to record, for each basic neighborhood N_s, the ordinal rank of $f_y \restriction N_s$ in $\leq^* \restriction \{f_z \restriction N_s : B(z)\}$. See [KM78B, 3C-1].

Using more sophisticated comparison games, one can combine the techniques for propagating scales under universal and existential real quantification, as well as existential ordinal quantification. This leads to the propagation of scales under various *game quantifiers*. We shall discuss these results in more detail in the next section.

Although the fake-sup method of propagating scales was invented in order to obtain optimally definable scales, one can show that under AD, the tree of the scale it produces is very often homogeneous. (The tree of any scale is the surjective image of $\mathbb{R}$, so it is too small to be homogeneous in **V**.) See [MS07], where it is also shown that the tree very often has the "generic codes" property of [KW07].

§3. **Individual papers.** We pass to an extended table of contents for the papers in the block to follow, together with pointers to some related results and literature. We also include a number of proof sketches. Some of these sketches will only make sense to readers with significant background knowledge. We have included references to fuller explanations in the literature when possible.

Notes on the theory of scales [KM78B].

This is a survey paper, written in 1971. It is still an excellent starting point for anyone seeking basic information regarding the construction and use of scales under determinacy hypotheses. It is truly remarkable how much of the descriptive set theory that is founded on large cardinals and determinacy emerged in the early years of the subject.

The paper begins in §2 – §4 with the inf and fake-sup constructions, and their corollaries regarding the scale property and uniformization in the projective hierarchy.

Theorem 3.1 (Moschovakis 1970). Assume all $\undertilde{\mathbf{\Delta}}^1_{2n}$ games are determined; then

(1) Π^1_{2n+1} and Σ^1_{2n+2} have the scale property, and hence
(2) every Π^1_{2n+1} (respectively Σ^1_{2n+2}) relation on $\mathbb{R}$ can be uniformized by a Π^1_{2n+1} (respectively Σ^1_{2n+2}) function.

In §6, the **projective ordinals**

$$\undertilde{\delta}^1_n := \sup\{\alpha : \alpha \text{ is the order type of a } \undertilde{\mathbf{\Delta}}^1_n \text{ prewellorder of } \mathbb{R}\}$$

are introduced. One can show that, assuming PD, any Π^1_{2n+1}-norm on a complete Π^1_{2n+1} set has length $\undertilde{\delta}^1_{2n+1}$; see [Mos80, 4C.14]. From the scale property for Π^1_{2n+1} one then gets that all Π^1_{2n+1} sets are $\undertilde{\delta}^1_{2n+1}$-Suslin, and thence that all Σ^1_{2n+2} sets are $\undertilde{\delta}^1_{2n+1}$-Suslin. (For $n = 0$, this reduces to the classical

Novikoff-Kondô result that all Σ^1_2 sets are ω_1-Suslin.) The size of the projective ordinals, both in inner models of AD, and in the full universe **V**, is therefore a very important topic. It is a classical result that $\boldsymbol{\delta}^1_1 = \omega_1$, while the size of the larger projective ordinals has been the subject of much later work, some of which will be collected in a block of papers in a later volume in this series.

§7 proves the Kunen-Martin theorem:

THEOREM 3.2 (Kunen, Martin). Every κ-Suslin wellfounded relation on $\mathbb{R}$ has rank $< \kappa^+$.

This basic result has important corollaries concerning the sizes of the projective ordinals. For example, because all Σ^1_{2n+2} sets are $\boldsymbol{\delta}^1_{2n+1}$-Suslin, we have that $\boldsymbol{\delta}^1_{2n+2} \leq (\boldsymbol{\delta}^1_{2n+1})^+$, and in particular, $\boldsymbol{\delta}^1_2 \leq \omega_2$.

§8 investigates the way in which Suslin representations yield ∞-Borel representations. It is shown that κ-Suslin sets are κ^{++}-Borel (i.e. can be built up from open sets using complementation and wellordered unions of length $< \kappa^{++}$). Of course, if CH holds, then *every* set of reals is a union of ω_1 singletons; the true content of the result of §8 lies in the fact that the κ^{++}-Borel representation is definable from the κ-Suslin representation. §8 also shows that, assuming PD, every $\boldsymbol{\Delta}^1_{2n+1}$ set is $\boldsymbol{\delta}^1_{2n+1}$-Borel. If $n = 0$, this is just Suslin's original theorem. In order to obtain a converse when $n > 0$, we must impose a definability restriction on our $\boldsymbol{\delta}^1_{2n+1}$-Borel representation, since again, it could be that every set of reals is $\omega_1 + 1$-Borel. One way to do that is to assume full AD, and Martin showed that indeed, assuming AD, every $\boldsymbol{\Delta}^1_{2n+1}$ set is $\boldsymbol{\delta}^1_{2n+1}$-Borel. So we have

THEOREM 3.3 (Martin, Moschovakis). Assume AD; then the $\boldsymbol{\Delta}^1_{2n+1}$ sets of reals are precisely the $\boldsymbol{\delta}^1_{2n+1}$-Borel sets.

See [Mos80, 7D.9]. This fully generalizes Suslin's 1917 theorem to the higher levels of the projective hierarchy.

§5 and §9 introduce inner models, obtained from Suslin representations, which have certain degrees of correctness. In §5, it is shown that for $n \geq 2$, there is a unique, minimal Σ^1_n-correct inner model M^*_n containing all the ordinals; the model is obtained by closing under constructibility and an optimally definable Skolem function for Σ^1_n. (Kechris and Moschovakis call this model M_n—not to be confused with $\mathbf{M}_n$; see below.) §9 considers the model $\mathbf{L}[T]$, where T is the tree of a Π^1_{2n+1} scale on a complete Π^1_{2n+1} set. These models have proved more important in later work than the M^*_n. It is shown that if $n = 0$, then $\mathbf{L}[T] = \mathbf{L}$; in particular, $\mathbf{L}[T]$ is independent of the Π^1_{2n+1} scale and complete set chosen. Moschovakis conjectured that $\mathbf{L}[T]$ is independent of these choices if $n > 0$ as well, and more vaguely, that it is a "correct higher level analog of **L**".

Becker's paper [Bec78] contains an excellent summary of what was known in 1977 about the models of §5 and §9. The independence conjecture, which