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Preface

Sometimes in mathematics a simple-looking observation opens up a new road to a fertile field. Such an observation was made independently by Garrett Birkhoff [25] and Hans Samelson [192], who remarked that one can use Hilbert's (projective) metric and the contraction mapping principle to prove some of the theorems of Perron and Frobenius concerning eigenvectors and eigenvalues of nonnegative matrices. This idea has been pivotal for the development of nonlinear Perron–Frobenius theory.

In the past few decades a number of strikingly detailed nonlinear extensions of Perron–Frobenius theory have been obtained. These results provide an extensive analysis of the eigenvectors and eigenvalues of various classes of order-preserving (monotone) nonlinear maps and give information about their iterative behavior and periodic orbits. Particular classes of order-preserving maps for which there exist nonlinear Perron–Frobenius theorems include sub-homogeneous maps, topical maps, and integral-preserving maps. The latter class of order-preserving maps can be regarded as a nonlinear generalization of column stochastic matrices, whereas topical maps generalize row stochastic matrices

The main purpose of this book is to give a systematic, self-contained introduction to nonlinear Perron–Frobenius theory and to provide a guide to various challenging open problems. We hope that it will be a stimulating source for non-experts to learn and appreciate this subject. To keep our task manageable, we restrict ourselves to **finite-dimensional** vector spaces, which allows us to avoid the use of sophisticated fixed-point theorems, the fixed-point index, and topological degree theory. The material in this book requires familiarity only with basic real analysis and topology, and is accessible to graduate students.

Classical Perron–Frobenius theory was developed in the early 1900s. In fact, Perron published his work [179, 180] on eigenvalues and eigenvectors of matrices with positive coefficients in 1907. His results were generalized by

Frobenius, in a series of papers [70–72] a few years later, to irreducible matrices with nonnegative coefficients. Their collective results and subsequent work are known today as Perron–Frobenius theory and are considered one of the most beautiful topics in linear algebra. The theory has numerous applications in probability theory, game theory, information theory, dynamical systems theory, mathematical biology, mathematical economics, and computer science.

In a seminal paper, Kreĭn and Rutman [117] placed the Perron–Frobenius theorem in the more general context of linear operators that leave a cone in a Banach space invariant. Their work has led to numerous studies of the spectral theory of positive linear operators on Banach spaces, including work by Bonsall [29–32] and Schaefer [193–195]. A detailed account can be found in Schaefer's book [196]. In their work Kreĭn and Rutman combined analytic methods with geometric ones. Among other methods they applied the Brouwer fixed-point theorem to find a positive eigenvector of a nonsingular matrix that leaves a cone invariant. Their geometric ideas are another source of inspiration for nonlinear Perron–Frobenius theory.

The book contains nine chapters and two appendices. The first four chapters contain preliminaries to Chapters 5, 6, 7, 8, and 9, which form the core of the book. The main objective of Chapter 1 is to introduce the reader to a variety of questions in nonlinear Perron–Frobenius theory. To this end we recall the classical results from Perron–Frobenius theory. Some of their proofs are given in Appendix B and are nonlinear in spirit. We also introduce various classes of nonlinear order-preserving maps for which there exist nonlinear Perron–Frobenius theorems and provide motivating examples.

Chapter 2 develops the relation between order-preserving maps and non-expansive maps. It shows how various classes of order-preserving maps are non-expansive under Hilbert's metric, Thompson's metric, or a polyhedral norm. At the heart of these results lies the observation by Birkhoff and Samelson. In addition, several results concerning the geometry and topology of Hilbert's and Thompson's metric spaces are collected.

Chapter 3 covers several useful topics on the iterative behavior of non-expansive maps including ω-limit sets, fixed-point theorems, horofunctions, and horoballs. It also contains a discussion of "Denjoy–Wolff type" theorems for fixed-point free non-expansive maps on metric spaces whose geometry resembles that of a hyperbolic space, which are due to Beardon [18, 19] and were further developed by Karlsson [99].

Chapter 4 focuses on the dynamics of sup-norm non-expansive maps. A characteristic property of sup-norm non-expansive maps is that either all orbits are unbounded or each orbit converges to a periodic orbit. Moreover, there exists an a-priori upper bound for the periods of their periodic points in terms of the dimension of the underlying space. These results are a key ingredient in the analysis of the iterative behavior of order-preserving subhomogeneous maps on polyhedral cones. In addition, the dynamics and periodic orbits of topical maps are discussed.

Chapter 5 deals with eigenvectors and the cone spectral radius of order-preserving homogeneous maps on closed cones in a finite-dimensional vector space. The cone spectrum and the cone spectral radius are analyzed. Among other results it is shown that there exists an eigenvector in the cone corresponding to the cone spectral radius. The chapter also treats the continuity problem of the cone spectral radius and discusses nonlinear generalizations of the classical Collatz–Wielandt formula for the spectral radius of nonnegative matrices.

Chapter 6 is mainly concerned with the question whether there exists an eigenvector in the interior of the cone for a given order-preserving homogeneous map. It appears that this problem is irreducibly difficult. Several general principles are discussed that are helpful when faced with this problem. These principles and their limitations are illustrated by analyzing particular order-preserving homogeneous maps involving means.

In Chapter 7 we illustrate how the results from Chapters 5 and 6 can be applied to finding solutions to various matrix scaling problems. We follow the fixed-point approach, as pioneered by Menon [143]. Among other matters we discuss the elegant solution, independently discovered by Sinkhorn and Knopp [206] and Brualdi, Parter, and Schneider [39], of the classic DAD problem: Given an $n \times n$ nonnegative matrix A, when do there exist positive diagonal matrices D and E such that DAE is doubly stochastic?

In Chapter 8 we derive a variety of results for order-preserving subhomogeneous maps on finite-dimensional cones. A central question is the behavior of the iterates of such maps. We provide a detailed analysis of the periodic orbits of order-preserving subhomogeneous maps on polyhedral cones. We also discuss "Denjoy–Wolff type" theorems for order-preserving homogeneous maps which do not have an eigenvector in the interior of the cone.

Chapter 9 is devoted to nonlinear Perron–Frobenius theorems for order-preserving integral-preserving maps on the standard positive cone. Such maps are non-expansive under the ℓ_1-norm. It is shown how the dynamics of order-preserving integral-preserving maps is related to the dynamics of lower semi-lattice homomorphisms. This connection yields a complete combinatorial characterization of the set of possible periods of periodic points of order-preserving integral-preserving maps in terms of periods of so-called admissible arrays. This characterization allows one to compute the set of possible periods of periodic points of order-preserving integral-preserving maps on the standard positive cone in finite time.

This book does not attempt to be an encyclopedic coverage of nonlinear Perron–Frobenius theory, even for finite-dimensional spaces. The expert reader may note that the following topics have been omitted: applications to the theory of ordinary differential equations [87,88,115,164,228], the cycle time problem [36, 51, 152], and the spectral theory of order-preserving convex maps [3, 4]. However, the authors believe that an understanding of the material in this book will leave the reader well equipped to master the existing literature on these topics.

Nonlinear Perron–Frobenius theory is related to monotone dynamical systems theory. In the theory of monotone dynamical systems one considers discrete and continuous dynamical systems that are strongly order-preserving or strongly monotone. Pioneering work in this field was done by Hirsch [87] who showed, among other results, that in a continuous time strongly monotone dynamical system almost all pre-compact orbits converge to the set of equilibrium points. For discrete time strongly monotone dynamical systems one has generic convergence to periodic orbits under appropriate conditions; see [182]. An extensive overview of these results was given by Hirsch and Smith [89]; see also Smith [207]. In monotone dynamical systems theory, emphasis is placed on strong monotonicity. If, however, one only assumes the dynamical system to be monotone, most of the theory is not applicable. In nonlinear Perron–Frobenius theory one usually considers discrete time dynamical systems that are only monotone (order-preserving), but satisfy an additional assumption such as subhomogeneity, additive homogeneity, or the integral-preserving condition. Another notable difference between the two theories is that in monotone dynamical systems theory one usually assumes the system to be differentiable. In nonlinear Perron–Frobenius theory, the discrete dynamical system is often only continuous. These different assumptions on the dynamical system require different methods and give the two theories a very different character. We hope that this book will also be a valuable addition to the existing literature on monotone dynamical systems theory.

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