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What is nonlinear Perron–Frobenius theory?

To get an impression of the contents of nonlinear Perron–Frobenius theory, it is useful to first recall the basics of classical Perron–Frobenius theory. Classical Perron–Frobenius theory concerns nonnegative matrices, their eigenvalues and corresponding eigenvectors. The fundamental theorems of this classical theory were discovered at the beginning of the twentieth century by Perron [179, 180], who investigated eigenvalues and eigenvectors of matrices with strictly positive entries, and by Frobenius [70–72], who extended Perron's results to irreducible nonnegative matrices. In the first section we discuss the theorems of Perron and Frobenius and some of their generalizations to linear maps that leave a cone in a finite-dimensional vector space invariant. The proofs of these classical results can be found in many books on matrix analysis, e.g., [15, 22, 73, 148, 202]. Nevertheless, in Appendix B we prove some of them once more using a combination of analytic, geometric, and algebraic methods. The geometric methods originate from work of Alexandroff and Hopf [8], Birkhoff [25], Kreĭn and Rutman [117], and Samelson [192] and underpin much of nonlinear Perron–Frobenius theory. Readers who are not familiar with these methods might prefer to first read Chapters 1 and 2 and Appendix B. Besides recalling the classical Perron–Frobenius theorems, we use this chapter to introduce some basic concepts and terminology that will be used throughout the exposition, and provide some motivating examples of classes of nonlinear maps to which the theory applies.

We emphasize that throughout the book we will always be working in a **finite-dimensional** real vector space V, unless we explicitly say otherwise.

1.1 Classical Perron–Frobenius theory

An $n \times n$ matrix $A = (a_{ij})$ is said to be *nonnegative* if $a_{ij} \geq 0$ for all i and j. It is called *positive* if $a_{ij} > 0$ for all i and j. Similarly, we call a vector $x \in \mathbb{R}^n$

nonnegative (or *positive*) if all its coordinates are nonnegative (or positive). The *spectrum* of A is given by

$$\sigma(A) = \{\lambda \in \mathbb{C} : Ax = \lambda x \text{ for some } x \in \mathbb{C}^n \setminus \{0\}\}.$$

Recall also that the *spectral radius* of A is given by

$$r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\},$$

and satisfies the equality $r(A) = \lim_{k\to\infty} \|A^k\|^{1/k}$. Notice that the limit is independent of the choice of the matrix norm, or norm on $\mathbb{R}^{n^2}$, as they are all equivalent; see Rudin [190]. The following result is due to Perron [180].

Theorem 1.1.1 (Perron) *If A is a positive matrix, then the following assertions hold:*

(i) $r(A) > 0$, and $r(A)$ is an algebraically simple eigenvalue of A and the corresponding normalized eigenvector v is unique and positive.
(ii) Any nonnegative eigenvector of A is a multiple of v.
(iii) For each eigenvalue $\lambda \in \sigma(A)$ with $\lambda \neq r(A)$ we have that $|\lambda| < r(A)$.

In a series of papers Frobenius [70–72] extended Perron's theorem to so-called irreducible matrices. An $n \times n$ nonnegative matrix $A = (a_{ij})$ is called *reducible* if $\{1, \ldots, n\}$ can be partitioned into two non-empty sets I and J such that $a_{ij} = 0$ for all $i \in I$ and $j \in J$. In other words, A is reducible if and only if there exists a permutation matrix P such that

$$P^T A P = \begin{bmatrix} B & C \\ O & D \end{bmatrix},$$

where B and D are square matrices and O is the zero matrix. A nonnegative matrix is said to be *irreducible* if it is not reducible. In particular, any nonnegative 1×1 matrix is irreducible. Frobenius proved the following generalization of Perron's result.

Theorem 1.1.2 (Perron–Frobenius) *If A is a nonnegative irreducible $n \times n$ matrix, then the following assertions hold:*

(i) $r(A)$ is an algebraically simple eigenvalue of A and the corresponding normalized eigenvector v is unique and positive. Moreover $r(A) > 0$, if $A \neq [0]$.
(ii) Any nonnegative eigenvector of A is a multiple of v.
(iii) If, in addition, A has exactly q eigenvalues λ with $|\lambda| = r(A)$, then these eigenvalues are given by $r(A)e^{2\pi ik/q}$ for $0 \leq k < q$.

The integer q in Theorem 1.1.2 is called the *index of cyclicity* of A.

Nonnegative matrices leave the cone of nonnegative vectors in $\mathbb{R}^n$ invariant. This is a crucial property of nonnegative matrices. In fact, a major part of the Perron–Frobenius theory can be generalized to linear maps that leave a cone in a vector space invariant. We will discuss this important fact in greater detail now. Let V be a finite-dimensional real vector space. A subset K of V is called a *cone* if it is convex, $\mu K \subseteq K$ for all $\mu \geq 0$, and $K \cap (-K) = \{0\}$. It is said to be a *closed* cone if it is a closed set in V with respect to the standard topology. Given a subset $S \subseteq V$, the interior, closure, and boundary of S with respect to the standard topology on V are denoted, respectively by $\mathrm{int}(S)$, $\mathrm{cl}(S)$, and ∂S. A cone is said to be *solid* if it has a non-empty interior. Basic examples of solid closed cones include the *standard positive cone*,

$$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for } 1 \leq i \leq n\},$$

and the *Lorentz cone*,

$$\Lambda_{n+1} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 - x_2^2 - \dots - x_{n+1}^2 \geq 0 \text{ and } x_1 \geq 0\}.$$

Other interesting examples arise in spaces of matrices. For example, if V is the space of real $n \times n$ symmetric matrices, then the set of positive-semidefinite matrices, $\Pi_n(\mathbb{R})$, is a solid closed cone in V.

We write V^* to denote the dual space of V and define the *dual cone* of K by

$$K^* = \{\varphi \in V^* : \varphi(x) \geq 0 \text{ for all } x \in K\}.$$

A *face* of a closed cone K in V is a non-empty convex subset F of K such that if $x, y \in K$ and $\lambda x + (1-\lambda)y \in F$ for some $0 < \lambda < 1$, then $x, y \in F$. Note that K itself and $\{0\}$ are faces. These two faces of K are called *improper faces*. All other faces of K are said to be *proper*. The relative interior of a convex set $C \subseteq V$, denoted relint (C), is the interior of C, regarded as a subset of the affine hull of C in V. It is known (see [186, theorem 6.2]) that $\mathrm{cl}(C)$ and relint (C) are non-empty and convex if C is a non-empty convex subset of V. We can use this to show that each face of a closed cone is closed. Indeed, given a face F of K, let z be a point in the relative interior of F. Now note that, for each $x \in \mathrm{cl}(F)$ with $x \neq z$, there exists $y \in \mathrm{cl}(F)$ such that z is in the relative interior of the straight-line segment from x to y, and hence $x \in F$.

A face F of K is called an *exposed face* if there exists $\varphi \in K^*$ such that $F = K \cap \{x \in V : \varphi(x) = 0\}$. In general not every face of a cone is exposed. However, each face of a polyhedral cone is exposed. A cone K in V is *polyhedral* if it is the intersection of finitely many closed half-spaces, i.e., there exists $\varphi_1, \dots, \varphi_m \in V^*$ such that

$$K = \{x \in V : \varphi_i(x) \geq 0 \text{ for all } 1 \leq i \leq m\}.$$

A face F of a polyhedral cone K is called a *facet* if $\dim(F) = \dim(K) - 1$. Here $\dim(F)$ denotes the dimension of the linear span of F. The following basic result from polyhedral geometry [201, section 8.4] will be useful in the sequel.

Lemma 1.1.3 *If $K \subseteq V$ is a polyhedral cone with N facets, then there exist N linear functionals $\psi_1, \ldots, \psi_N$ such that*

$$K = \{x \in V : \psi_i(x) \geq 0 \textit{ for all } 1 \leq i \leq N\} \cap \operatorname{span}(K)$$

and each linear functional ψ_i corresponds to a unique facet of K.

The functionals $\psi_1, \ldots, \psi_N$ in Lemma 1.1.3 are called *facet-defining functionals* of the polyhedral cone. For example, $\mathbb{R}^n_+$ has 2^n faces, $F_I = \{x \in \mathbb{R}^n_+ : x_i > 0$ if and only if $i \in I\}$ for $I \subseteq \{1, \ldots, n\}$, and n facets corresponding to those I with $|I| = n - 1$.

A cone K in V induces a partial ordering $\leq_K$ on V by

$$x \leq_K y \text{ if } y - x \in K.$$

If K has a non-empty interior, then we write $x \ll_K y$ if $y - x \in \operatorname{int}(K)$. We also write $x <_K y$ if $x \leq_K y$ and $x \neq y$. We simply write $x \leq y$, $x < y$, and $x \ll y$ if K is clear from the context. In the special case where $K = \mathbb{R}^n_+$ we note that $x \leq y$ if and only if $x_i \leq y_i$ for all $1 \leq i \leq n$. Note that a linear map $A : V \to V$ leaves a cone K invariant if and only if $0 \leq_K Ax$ for all $0 \leq_K x$ in V.

There is a natural way to generalize the concept of irreducibility to linear maps that leave a solid closed cone invariant.

Definition 1.1.4 A linear map $A : V \to V$ that leaves a solid closed cone K invariant is said to be *irreducible* if no proper face of K is left invariant by A.

It is a simple exercise to show that a nonnegative matrix A is irreducible in the sense of Definition 1.1.4 if and only if it is irreducible in the usual sense. Another equivalent way to define the notion of irreducibility is given in the following proposition.

Proposition 1.1.5 *A linear map $A : V \to V$ that leaves a solid closed cone K invariant is irreducible if and only if $(\lambda I - A)^{-1}(K \setminus \{0\}) \subseteq \operatorname{int}(K)$ for some $\lambda > r(A)$.*

Proof Suppose, for the sake of contradiction, that A is irreducible and there exists $z \in K \setminus \{0\}$ such that $(\lambda I - A)^{-1}z \notin \operatorname{int}(K)$ for all $\lambda > r(A)$. Write $u = (\lambda I - A)^{-1}z$. By the Hahn–Banach separation theorem [186, Theorem 11.6] there exists $\varphi \in K^* \setminus \{0\}$ such that $\varphi(u) = 0$ and $\varphi(x) > 0$ for all $x \in \operatorname{int}(K)$. Define $\psi \in V^*$ by

$$\psi(x) = \varphi((\lambda I - A)^{-1}x) = \sum_{k=0}^{\infty} \lambda^{-k-1}\varphi(A^k x),$$

and remark that $\psi(x) > 0$ for all $x \in \operatorname{int}(K)$. Obviously $\psi(z) = 0$ and hence $F = \{x \in K : \psi(x) = 0\}$ is a proper (exposed) face of K. Moreover, for each $x \in F$ we have that

$$0 \leq \psi(Ax) = \varphi((\lambda I - A)^{-1}(Ax)) = \sum_{k=0}^{\infty} \lambda^{-k-1}\varphi(A^{k+1}x)$$
$$= \lambda \sum_{k=1}^{\infty} \lambda^{-k-1}\varphi(A^k x) \leq \lambda\psi(x) = 0.$$

This implies that $\psi(Ax) = 0$ for all $x \in F$. Thus, A leaves the proper face F invariant, which is impossible.

To establish the equivalence, suppose that for some $\lambda > r(A)$ we have that $(\lambda I - A)^{-1}z \in \operatorname{int}(K)$ for all $z \in K \setminus \{0\}$. Now if there exists a proper face F of K such that $A(F) \subseteq F$, then it follows from the identity, $(\lambda I - A)^{-1} = \sum_{k=0}^{\infty} \lambda^{-k-1}A^k$, that $(\lambda I - A)^{-1}(F) \subseteq F$, which is impossible. □

Note that the proof of Proposition 1.1.5 also shows that a linear map A is irreducible if and only if it leaves no proper exposed face of K invariant.

As mentioned earlier the theorems of Perron and Frobenius can be generalized to linear maps that leave a cone invariant. This important observation was made by Kreĭn and Rutman in their pioneering work [117], in which they studied linear operators that leave a cone in a possibly infinite-dimensional normed space invariant. As we are only concerned with finite-dimensional spaces in this book, we give a finite-dimensional version of their theorem here.

Theorem 1.1.6 (Kreĭn–Rutman) *If $A : V \to V$ is a linear map that leaves a solid closed cone K invariant, then $r(A)$ is an eigenvalue of A and $r(A)$ has an eigenvector in K.*

Just as for the Perron–Frobenius Theorem 1.1.2 we have uniqueness of the eigenvector if A is irreducible.

Theorem 1.1.7 *If $A : V \to V$ is an irreducible linear map that leaves a solid closed cone K invariant, then the following assertions hold:*

(i) $r(A)$ is an algebraically simple eigenvalue of A and the corresponding normalized eigenvector v is unique and lies in $\operatorname{int}(K)$. *Furthermore, $r(A) > 0$, if $A \neq [0]$.*
(ii) Any nonnegative eigenvector of A is a multiple of v.

The third part of the Perron–Frobenius Theorem 1.1.2 cannot be extended to arbitrary closed cones. Simply consider the Lorentz cone Λ_3 and the linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Clearly $A(\Lambda_3) \subseteq \Lambda_3$ and $\sigma(A) = \{1, e^{\pm i\vartheta}\}$. If ϑ is an irrational multiple of 2π, then $e^{i\vartheta}$ is not a root of unity. The third part of the Perron–Frobenius Theorem 1.1.2 can, however, be generalized to polyhedral cones (see [117, section 8]).

Theorem 1.1.8 *If $A : V \to V$ is a linear map that leaves a solid polyhedral cone K with N facets invariant, then for each $\lambda \in \sigma(A)$ with $|\lambda| = r(A)$ there exists $1 \leq q \leq N$ such that*

$$\lambda^q = r(A)^q.$$

Theorem 1.1.8 can be used to prove the following result concerning the iterative behavior of linear maps that leave a polyhedral cone invariant; see Appendix B for details.

Theorem 1.1.9 *If $A : V \to V$ is a linear map that leaves a solid polyhedral cone K with N facets invariant, then there exists an integer $p \geq 1$ such that*

$$\lim_{k\to\infty} A^{kp}x$$

exists for each $x \in K$ with $(\|A^k x\|)_k$ bounded. Moreover, p is the order of a permutation on N letters.

For linear maps $A : V \to V$ the condition that $A(K) \subseteq K$ is equivalent to $x \leq_K y$ implies $Ax \leq_K Ay$ for $x, y \in V$. Given cones $K \subseteq V$ and $K' \subseteq V'$, a map $f : X \to X'$, with $X \subseteq V$ and $X' \subseteq V'$, is said to be *order-preserving* if $x \leq_K y$ implies $f(x) \leq_{K'} f(y)$ for $x, y \in X$. It is said to be *strongly order-preserving* if $x <_K y$ implies $f(x) \ll_{K'} f(y)$. Moreover, we say that $f : X \to X'$ is *order-reversing* if $x \leq_K y$ implies $f(y) \leq_{K'} f(x)$ for $x, y \in X$.

Nonlinear Perron–Frobenius theory is primarily concerned with order-preserving maps and treats questions like:

- Is there a sensible definition of the spectral radius for order-preserving maps $f : K \to K$, and does there exist a corresponding eigenvector?
- When does an order-preserving map have an eigenvector in the interior of the cone, and when is it unique?

- How do the orbits

$$x,\ f(x),\ f^2(x) = f(f(x)),\ f^3(x) = f(f(f(x))), \ldots$$

  of an order-preserving map f behave in the long term?
- When does an order-preserving map have the property that every bounded orbit converges to a periodic orbit?

These questions arise in numerous applications and lie at the heart of nonlinear Perron–Frobenius theory. They motivate much of the material discussed in this book. As we shall see, strikingly detailed answers exist for a variety of classes of order-preserving map.

1.2 Cones and partial orderings

Given a cone $K \subseteq V$ and $x, y \in V$ it is often useful to use the following result to decide whether $x \leq_K y$.

Lemma 1.2.1 *If $K \subseteq V$ is a closed cone, then $x \leq_K y$ if and only if $\varphi(x) \leq \varphi(y)$ for all $\varphi \in K^*$.*

Proof Note that if $x \not\leq_K y$, then $y - x \notin K$. By the Hahn–Banach separation theorem [186, theorem 11.4] there exist $\alpha \in \mathbb{R}$ and $\psi \in V^*$ such that $\psi(y-x) < \alpha$ and $\psi(v) > \alpha$ for all $v \in K$. Remark that, as $\lambda v \in K$ for all $\lambda \geq 0$ and $v \in K$, $\psi(v) \geq 0$ for all $v \in K$, and hence $\psi \in K^*$. Also note that $\psi(0) = 0$, so that $\alpha < 0$. Thus, $\psi(y) < \psi(x)$. The other implication is trivial. □

We say that $y \in K$ *dominates* $x \in V$ if there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha y \leq x \leq \beta y$. This notion yields an equivalence relation, $\sim_K$, on K by $x \sim_K y$ if x dominates y and y dominates x. It is easy to verify that $x \sim_K y$ if and only if there exist $0 < \alpha \leq \beta$ such that $\alpha y \leq x \leq \beta y$. The equivalence classes in K are called *parts* of the cone, and we write $[x]$ to denote the part of K containing x. The set of all parts of K is denoted by $\mathcal{P}(K)$. For instance, the parts of $\mathbb{R}^2_+$ are $\{0\}$, $\mathrm{int}(\mathbb{R}^2_+)$, $\{(x_1, 0) \in \mathbb{R}^2_+ : x_1 > 0\}$ and $\{(0, x_2) \in \mathbb{R}^2_+ : x_2 > 0\}$. The parts of Λ_3 are given by $\{0\}$, $\mathrm{int}(\Lambda_3)$, and $\{\lambda(1, \cos\vartheta, \sin\vartheta) : \lambda > 0\}$ for $0 \leq \vartheta < 2\pi$.

Lemma 1.2.2 *If $K \subseteq V$ is a closed cone, then the parts of K are precisely the relative interiors of the faces of K.*

Proof It is known (see [186, theorem 8.2]) that the relative interiors of the faces of K partition K. Let $[x]$ be the part of x in K and F_x be the face of K with x in its relative interior. Note that F_x is itself a closed cone. Suppose that y is in the relative interior of F_x. Then there exists $\alpha > 0$ such that $y - \alpha x \in F_x$ and $x - \alpha y \in F_x$. As $F_x \subseteq K$, we deduce that $\alpha x \leq y \leq x/\alpha$, and hence $x \sim_K y$.

On the other hand, if $y \in K$ is such that $x \sim_K y$, then for $\delta > 0$ sufficiently small, $x - \delta y \sim_K y$ and $y - \delta x \sim_K x$. As $x \sim_K y$ implies $\lambda x \sim_K \mu y$ for all $\lambda, \mu > 0$, we deduce for $\varepsilon > 0$ sufficiently small that

$$x_\varepsilon = (1+\varepsilon)x - \varepsilon y \sim_K y \quad \text{and} \quad y_\varepsilon = (1+\varepsilon)y - \varepsilon x \sim_K x.$$

Note that $x = a x_\varepsilon + b y_\varepsilon$ for $a = (1+\varepsilon)/(1+2\varepsilon)$ and $b = \varepsilon/(1+2\varepsilon)$. As $x \in F_x$, we deduce that x_ε and y_ε in F_x, since x is in the relative interior of F_x. As $y = \lambda y_\varepsilon + (1-\lambda)x$ for $\lambda = 1/(1+\varepsilon)$, we conclude that y is in the relative interior of F_x. □

It follows from Lemma 1.2.2 that a polyhedral cone has finitely many parts. The combinatorics of the parts of a polyhedral cone will play an important role in the sequel, especially in Chapter 8. Given a polyhedral cone K with N facets and facet-defining functionals $\psi_1, \ldots, \psi_N$ define for each $x \in K$ the set $I_x = \{i : \psi_i(x) > 0\}$. Likewise, if P is a part of K we let $I(P) = \{i : \psi_i(x) > 0 \text{ for some } x \in P\}$.

The set of parts $\mathcal{P}(K)$ has a natural partial ordering $\trianglelefteq$ given by $P \trianglelefteq Q$ if there exist $x \in P$ and $y \in Q$ such that y dominates x. Equivalently, $P \trianglelefteq Q$ if for each $x \in P$ and each $y \in Q$ we have that y dominates x.

Lemma 1.2.3 *If $K \subseteq V$ is a polyhedral cone with N facets, then the following assertions hold:*

(i) $I_x = I_y$ if and only if $x \sim_K y$.
(ii) For $P \in \mathcal{P}(K)$ we have

$$P = \{x \in K : \psi_i(x) > 0 \textit{ if and only if } i \in I(P)\}.$$

(iii) $P \trianglelefteq Q$ if and only if $I(P) \subseteq I(Q)$.
(iv) $|\mathcal{P}(K)| \leq 2^N$.

Proof Let $\psi_1, \ldots, \psi_N \in K^*$ be the facet-defining functionals of K. By Lemma 1.1.3 we know that $x \leq_K y$ is equivalent to $\psi_i(x) \leq \psi_i(y)$ for all i. This implies that $I_x \subseteq I_y$ if and only if y dominates x. Therefore $x \sim_K y$ is equivalent to $I_x = I_y$. It also shows that $x \in P$ if and only if $I_x = I(P)$, which proves the second assertion.

Now suppose that $P \trianglelefteq Q$. Then there exist $x \in P$ and $y \in Q$ such that y dominates x. So, $I(P) = I_x \subseteq I_y = I(Q)$. On the other hand, if $I(P) \subseteq I(Q)$, then for $x \in P$ and $y \in Q$ we know that there exists $\beta > 0$ such that $\psi_i(x) \leq \beta\psi_i(y)$ for all i. Thus, by Lemma 1.1.3 we find that y dominates x and hence $P \trianglelefteq Q$. The final assertion is a direct consequence of the third one. □

The shape of the cone plays a fundamental role in nonlinear Perron–Frobenius theory. A distinguished class of cones is the set of so-called strictly convex cones. A closed cone $K \subseteq V$ is called *strictly convex* if, for all $x, y \in \partial K \setminus \{0\}$, with $x \neq \alpha y$ for all $\alpha > 0$, we have that

$$\{\lambda x + (1-\lambda)y : 0 < \lambda < 1\} \subseteq \mathrm{int}(K).$$

The Lorentz cone, Λ_{n+1}, is a prime example of a strictly convex cone for $n \geq 2$. In the sequel it will become clear that there is a marked contrast in the theory between polyhedral cones, strictly convex cones, and cones that are neither polyhedral nor strictly convex. An important example of the latter type is the cone of positive-semidefinite matrices $\Pi_n(\mathbb{R})$ for $n \geq 3$.

The partial ordering induced by the standard positive cone is particularly nice. To fully appreciate this we need a few more definitions. Let K be a cone in V. If $S \subseteq V$ and $u \in V$ is such that $s \leq_K u$ for all $s \in S$, we say that u is an *upper bound* of S. If, in addition, each upper bound $v \in V$ of S satisfies $u \leq_K v$, then we call u the *supremum* of S and we write $u = \sup(S)$. In a similar way lower bounds and the *infimum* of S can be defined. Note that if $K = \mathbb{R}^n_+$, then $\sup(x, y)$ exists and satisfies $\sup(x, y)_i = \max\{x_i, y_i\}$ for all i. Likewise, $\inf(x, y)$ exists and satisfies $\inf(x, y)_i = \min\{x_i, y_i\}$ for all i. In that case we write $x \wedge y = \inf(x, y)$ and $x \vee y = \sup(x, y)$, so $(x \wedge y)_i = \min\{x_i, y_i\}$ and $(x \vee y)_i = \max\{x_i, y_i\}$ for $1 \leq i \leq n$. The operations $\wedge$ and $\vee$ turn $\mathbb{R}^n$ into a vector lattice.

In general cones $\sup(x, y)$ and $\inf(x, y)$ need not exist for all $x, y \in V$. Cones for which $\sup(x, y)$ and $\inf(x, y)$ exist for all $x, y \in V$ are called *minihedral*. It is known that a solid closed minihedral cone is essentially equal to $\mathbb{R}^n_+$. More precisely, it was shown in [117, 151] that a solid closed cone K in an n-dimensional real vector space V is minihedral if and only if K is *simplicial*, i.e., there exist n linearly independent vectors $x^1, \ldots, x^n \in V$ such that

$$K = \{x \in V : x = \sum_i \alpha_i x^i \text{ where } \alpha_i \geq 0 \text{ for } 1 \leq i \leq n\}.$$

We conclude this section with some basic facts about dual cones and the normality property. Recall that the dual cone of $K \subseteq V$ is given by $K^* = \{\varphi \in V^* : \varphi(x) \geq 0 \text{ for all } x \in K\}$. In general K^* need not be a cone. However, the following is true.

Lemma 1.2.4 *If $K \subseteq V$ is a solid closed cone, then K^* is a solid closed cone. Moreover, if $\varphi \in \mathrm{int}(K^*)$, then $\varphi(x) > 0$ for all $x \in K$ with $x \neq 0$, and*

$$\Sigma_\varphi = \{x \in K : \varphi(x) = 1\}$$

is a compact convex subset of V.

Proof Clearly K^* is convex and $\lambda K^* \subseteq K^*$ for all $\lambda \geq 0$. Suppose that φ and $-\varphi$ are both in K^*, and $\varphi \neq 0$. Then $\varphi(x) \geq 0$ and $\varphi(x) \leq 0$ for all $x \in K$. This implies that $K \subseteq \{x \in V : \varphi(x) = 0\}$, which contradicts $\mathrm{int}(K) \neq \emptyset$.

Let $K^* - K^* = \{\varphi - \psi : \varphi, \psi \in K^*\}$, which is the smallest subspace of V^* containing K^*. For the sake of contradiction, suppose that $\mathrm{int}(K^*) = \emptyset$. We claim that $K^* - K^* \neq V^*$ in that case. Indeed, if $K^* - K^* = V^*$, then there exist linearly independent functionals in $\varphi_1, \ldots, \varphi_n \in K^*$, where $n = \dim(V^*)$. This implies that the convex hull of $\{0, \varphi_1, \ldots, \varphi_n\}$ is an n-dimensional simplex in K^* and hence $\mathrm{int}(K^*) \neq \emptyset$. Thus $K^* - K^* \neq V^*$, and hence there exists $z \in V \setminus \{0\}$ such that $K^* - K^* \subseteq \{\varphi \in V^* : \varphi(z) = 0\}$. It now follows from Lemma 1.2.1 that $z \in K$ and $-z \in K$, which is impossible, as K is a cone.

Let $\varphi \in \mathrm{int}(K^*)$ and suppose that $\varphi(x) = 0$ for some $x \in K$ with $x \neq 0$. As K is a cone, $-x \notin K$. By the Hahn–Banach separation theorem [186, theorem 11.4] there exists $\alpha \in \mathbb{R}$ and $\psi_x \in K^*$ such that $\psi_x(-x) < \alpha$ and $\psi_x(v) > \alpha$ for all $v \in K$. Since $\lambda v \in K$ for all $\lambda \geq 0$ and $v \in K$, $\psi_x(v) \geq 0$, so that $\psi_x \in K^*$ and $\alpha < 0$. This implies that $-\psi_x(x) < \alpha < 0$. As $\varphi \in \mathrm{int}(K^*)$ there exists $\varepsilon > 0$ such that $\varphi - \varepsilon\psi_x \in \mathrm{int}(K^*)$. So, $\varphi(x) - \varepsilon\psi_x(x) < 0$, which contradicts Lemma 1.2.1.

Let $\|\cdot\|$ be any norm on V and $S = \{x \in K : \|x\| = 1\}$, which is a compact set, as K is closed. Since $\varphi : K \to \mathbb{R}$ is continuous and strictly positive on S, there exists $\delta > 0$ such that $\varphi(y) \geq \delta$ for all $y \in S$. Thus, for each $x \in \Sigma_\varphi$ we have that $\|x\| \leq \delta^{-1}$, which shows that Σ_φ is bounded. Obviously Σ_φ is closed and convex, and hence compact, since V is finite-dimensional. □

If $K \subseteq V$ is a cone and V is equipped with a norm $\|\cdot\|$, we say that K is *normal* if there exists a constant $\delta > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \delta\|y\|$. The infimum of all such $\delta > 0$ is called the *normality constant*. We call $\|\cdot\|$ a *monotone norm* for K if the normality constant is equal to 1. It is a basic fact that every closed cone in a finite-dimensional normed space is normal.

Lemma 1.2.5 *Every closed cone K in $(V, \|\cdot\|)$ is normal.*

Proof For the sake of contradiction suppose that there exist sequences (x_k) and (y_k) in K such that $x_k \leq y_k$ for all $k \geq 1$ and

$$\lim_{k\to\infty} \frac{\|y_k\|}{\|x_k\|} = 0.$$

By rescaling x_k and y_k we may assume $\|x_k\| = 1$ and $x_k \leq y_k$ for all $k \geq 1$. This implies that $y_k \to 0$ as $k \to \infty$. Since V is finite-dimensional, we may further assume, after taking a subsequence, that (x_k) converges to some $x \in K$