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To my family: Tania, Joseph, & Kate.

Contents

Preface

The main purpose of this book is to bring together a number of results concerning the embedding of 'finite-dimensional' compact sets into Euclidean spaces, where an 'embedding' of a metric space (X, ϱ) into $\mathbb{R}^n$ is to be understood as a homeomorphism from X onto its image. A secondary aim is to present, alongside such 'abstract' embedding theorems, more concrete embedding results for the finite-dimensional attractors that have been shown to exist in many infinite-dimensional dynamical systems.

In addition to its summary of embedding results, the book also gives a unified survey of four major definitions of dimension (Lebesgue covering dimension, Hausdorff dimension, upper box-counting dimension, and Assouad dimension). In particular, it provides a more sustained exposition of the properties of the box-counting dimension than can be found elsewhere; indeed, the abstract results for sets with finite box-counting dimension are those that are taken further in the second part of the book, which treats finite-dimensional attractors.

While the various measures of dimension discussed here find a natural application in the theory of fractals, this is not a book about fractals. An example to which we will return continually is an orthogonal sequence in an infinite-dimensional Hilbert space, which is very far from being a 'fractal'. In particular, this class of examples can be used to show the sharpness of three of the embedding theorems that are proved here.

My models have been the classic text of Hurewicz & Wallman (1941) on the topological dimension, and of course Falconer's elegant 1985 tract which concentrates on the Hausdorff dimension (and Hausdorff measure). It is a pleasure to acknowledge formally my indebtedness to Hunt & Kaloshin's 1999 paper 'Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces'. It has had a major influence on my own research over the last ten years, and one could view this book as an extended exploration of the ramifications of the approach that they adopted there.

My interest in abstract embedding results is related to the question of whether one can reproduce the dynamics on a finite-dimensional attractor using a finite-dimensional system of ordinary differential equations (see Chapter 10 of Eden, Foias, Nicolaenko, & Temam (1994), or Chapter 16 of Robinson (2001), for example). However, there are still only partial results in this direction, so this potential application is not treated here; for an up-to-date discussion see the paper by Pinto de Moura, Robinson, & Sánchez-Gabites (2010).

I started writing this book while I was a Royal Society University Research Fellow, and many of the results here derive from work done during that time. I am currently supported by an EPSRC Leadership Fellowship, Grant EP/G007470/1. I am extremely grateful to both the Royal Society and to the EPSRC for their support.

I would like to thank Alexandre Carvalho, Peter Friz, Igor Kukavica, José Langa, Eric Olson, Eleonora Pinto de Moura, and Alejandro Vidal López, all of whom have had a hand in material that is presented here. In particular, Eleonora was working on closely-related problems for her doctoral thesis during most of the time that I was writing this book, and our frequent discussions have shaped much of the content and my approach to the material. I had comments on a draft version of the manuscript from Witold Sadowski, Jaime Sánchez-Gabites, and Nicholas Sharples: I am extremely grateful for their helpful and perceptive comments. David Tranah, Clare Dennison, and Emma Walker at Cambridge University Press have been most patient as one deadline after another was missed and extended; that one was finally met (nearly) is due in large part to a kind invitation from Marco Sammartino to Palermo, where I gave a series of lectures on some of the material in this book in November 2009.

Many thanks to my parents and to my mother-in-law; in addition to all their other support, their many days with the children have made this work possible. Finally, of course, thanks to Tania, my wife, and our children Joseph and Kate, who make it all worthwhile; this book is dedicated to them.