

Cambridge University Press

978-0-521-89805-8 - Dimensions, Embeddings, and Attractors

James C. Robinson

Frontmatter

[More information](#)

CAMBRIDGE TRACTS IN MATHEMATICS

General Editors

B. BOLLOBÁS, W. FULTON, A. KATOK, F. KIRWAN,  
P. SARNAK, B. SIMON, B. TOTARO

---

**186 Dimensions, Embeddings, and Attractors**

Cambridge University Press

978-0-521-89805-8 - Dimensions, Embeddings, and Attractors

James C. Robinson

Frontmatter

[More information](#)

---

# Dimensions, Embeddings, and Attractors

JAMES C. ROBINSON

*University of Warwick*



**CAMBRIDGE**  
UNIVERSITY PRESS

Cambridge University Press  
 978-0-521-89805-8 - Dimensions, Embeddings, and Attractors  
 James C. Robinson  
 Frontmatter  
[More information](#)

CAMBRIDGE UNIVERSITY PRESS  
 Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,  
 São Paulo, Delhi, Dubai, Tokyo, Mexico City

Cambridge University Press  
 The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)  
 Information on this title: [www.cambridge.org/9780521898058](http://www.cambridge.org/9780521898058)

© J. C. Robinson 2011

This publication is in copyright. Subject to statutory exception  
 and to the provisions of relevant collective licensing agreements,  
 no reproduction of any part may take place without the written  
 permission of Cambridge University Press.

First published 2011

Printed in the United Kingdom at the University Press, Cambridge

*A catalogue record for this publication is available from the British Library*

*Library of Congress Cataloging in Publication data*  
 Robinson, James C. (James Cooper), 1969–  
 Dimensions, Embeddings, and Attractors / James C. Robinson.  
 p. cm. – (Cambridge Tracts in Mathematics ; 186)  
 Includes bibliographical references and index.  
 ISBN 978-0-521-89805-8 (hardback)  
 1. Dimension theory (Topology) 2. Attractors (Mathematics)  
 3. Topological imbeddings. I. Title. II. Series.  
 QA611.3.R63 2011  
 515'.39 – dc22 2010042726

ISBN 978-0-521-89805-8 Hardback

---

Cambridge University Press has no responsibility for the persistence or  
 accuracy of URLs for external or third-party internet websites referred to in  
 this publication, and does not guarantee that any content on such websites is,  
 or will remain, accurate or appropriate.

---

Cambridge University Press  
978-0-521-89805-8 - Dimensions, Embeddings, and Attractors  
James C. Robinson  
Frontmatter  
[More information](#)

---

To my family: Tania, Joseph, & Kate.

## Contents

---

<i>Preface</i>	<i>page xi</i>
<b>Introduction</b>	<b>1</b>
PART I: FINITE-DIMENSIONAL SETS	
<b>1 Lebesgue covering dimension</b>	<b>7</b>
1.1 Covering dimension	8
1.2 The covering dimension of $I_n$	10
1.3 Embedding sets with finite covering dimension	12
1.4 Large and small inductive dimensions	17
Exercises	18
<b>2 Hausdorff measure and Hausdorff dimension</b>	<b>20</b>
2.1 Hausdorff measure and Lebesgue measure	20
2.2 Hausdorff dimension	23
2.3 The Hausdorff dimension of products	25
2.4 Hausdorff dimension and covering dimension	26
Exercises	29
<b>3 Box-counting dimension</b>	<b>31</b>
3.1 The definition of the box-counting dimension	31
3.2 Basic properties of the box-counting dimension	33
3.3 Box-counting dimension of products	35
3.4 Orthogonal sequences	36
Exercises	39
<b>4 An embedding theorem for subsets of <math>\mathbb{R}^N</math> in terms of the upper box-counting dimension</b>	<b>41</b>

<b>5</b>	<b>Prevalence, probe spaces, and a crucial inequality</b>	<b>47</b>
	5.1 Prevalence	47
	5.2 Measures based on sequences of linear subspaces	49
	Exercises	56
<b>6</b>	<b>Embedding sets with <math>d_H(X - X)</math> finite</b>	<b>57</b>
	6.1 No linear embedding is possible when $d_H(X)$ is finite	58
	6.2 Embedding sets with $d_H(X - X)$ finite	60
	6.3 No modulus of continuity is possible for $L^{-1}$	62
<b>7</b>	<b>Thickness exponents</b>	<b>64</b>
	7.1 The thickness exponent	65
	7.2 Lipschitz deviation	67
	7.3 Dual thickness	69
	Exercises	73
<b>8</b>	<b>Embedding sets of finite box-counting dimension</b>	<b>75</b>
	8.1 Embedding sets with Hölder continuous parametrisation	75
	8.2 Sharpness of the Hölder exponent	77
	Exercises	81
<b>9</b>	<b>Assouad dimension</b>	<b>83</b>
	9.1 Homogeneous spaces and the Assouad dimension	83
	9.2 Assouad dimension and products	86
	9.3 Orthogonal sequences	88
	9.4 Homogeneity is not sufficient for a bi-Lipschitz embedding	91
	9.5 Almost bi-Lipschitz embeddings	94
	9.6 Sharpness of the logarithmic exponent	99
	9.7 Consequences for embedding compact metric spaces	100
	Exercises	100
<b>PART II: FINITE-DIMENSIONAL ATTRACTORS</b>		
<b>10</b>	<b>Partial differential equations and nonlinear semigroups</b>	<b>105</b>
	10.1 Nonlinear semigroups and attractors	105
	10.2 Sobolev spaces and fractional power spaces	106
	10.3 Abstract semilinear parabolic equations	108
	10.4 The two-dimensional Navier–Stokes equations	109
	Exercises	113

<b>11</b>	<b>Attracting sets in infinite-dimensional systems</b>	<b>115</b>
11.1	Global attractors	115
11.2	Existence of the global attractor	115
11.3	Example 1: semilinear parabolic equations	118
11.4	Example 2: the two-dimensional Navier–Stokes equations	119
	Exercises	121
<b>12</b>	<b>Bounding the box-counting dimension of attractors</b>	<b>123</b>
12.1	Coverings of $T[B(0, 1)]$ via finite-dimensional approximations	125
12.2	A dimension bound when $Df \in \mathcal{L}_{\lambda/2}(\mathcal{B})$ , $\lambda < \frac{1}{2}$	129
12.3	Finite dimension when $Df \in \mathcal{L}_1(X)$	130
12.4	Semilinear parabolic equations in Hilbert spaces	130
	Exercises	132
<b>13</b>	<b>Thickness exponents of attractors</b>	<b>136</b>
13.1	Zero thickness	136
13.2	Zero Lipschitz deviation	138
	Exercises	143
<b>14</b>	<b>The Takens Time-Delay Embedding Theorem</b>	<b>145</b>
14.1	The finite-dimensional case	145
14.2	Periodic orbits and the Lipschitz constant for ordinary differential equations	152
14.3	The infinite-dimensional case	154
14.4	Periodic orbits and the Lipschitz constant for semilinear parabolic equations	156
	Exercises	158
<b>15</b>	<b>Parametrisation of attractors via point values</b>	<b>160</b>
15.1	Real analytic functions and the order of vanishing	161
15.2	Dimension and thickness of $\mathcal{A}$ in $C^r(\Omega, \mathbb{R}^d)$	163
15.3	Proof of Theorem 15.1	165
15.4	Applications	167
	Exercises	169
	<i>Solutions to exercises</i>	170
	<i>References</i>	196
	<i>Index</i>	202

Cambridge University Press

978-0-521-89805-8 - Dimensions, Embeddings, and Attractors

James C. Robinson

Frontmatter

[More information](#)

---

## Preface

---

The main purpose of this book is to bring together a number of results concerning the embedding of ‘finite-dimensional’ compact sets into Euclidean spaces, where an ‘embedding’ of a metric space  $(X, \varrho)$  into  $\mathbb{R}^n$  is to be understood as a homeomorphism from  $X$  onto its image. A secondary aim is to present, alongside such ‘abstract’ embedding theorems, more concrete embedding results for the finite-dimensional attractors that have been shown to exist in many infinite-dimensional dynamical systems.

In addition to its summary of embedding results, the book also gives a unified survey of four major definitions of dimension (Lebesgue covering dimension, Hausdorff dimension, upper box-counting dimension, and Assouad dimension). In particular, it provides a more sustained exposition of the properties of the box-counting dimension than can be found elsewhere; indeed, the abstract results for sets with finite box-counting dimension are those that are taken further in the second part of the book, which treats finite-dimensional attractors.

While the various measures of dimension discussed here find a natural application in the theory of fractals, this is not a book about fractals. An example to which we will return continually is an orthogonal sequence in an infinite-dimensional Hilbert space, which is very far from being a ‘fractal’. In particular, this class of examples can be used to show the sharpness of three of the embedding theorems that are proved here.

My models have been the classic text of Hurewicz & Wallman (1941) on the topological dimension, and of course Falconer’s elegant 1985 tract which concentrates on the Hausdorff dimension (and Hausdorff measure). It is a pleasure to acknowledge formally my indebtedness to Hunt & Kaloshin’s 1999 paper ‘Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces’. It has had a major influence on my own research over the last ten years, and one could view this book as an extended exploration of the ramifications of the approach that they adopted there.



My interest in abstract embedding results is related to the question of whether one can reproduce the dynamics on a finite-dimensional attractor using a finite-dimensional system of ordinary differential equations (see Chapter 10 of Eden, Foias, Nicolaenko, & Temam (1994), or Chapter 16 of Robinson (2001), for example). However, there are still only partial results in this direction, so this potential application is not treated here; for an up-to-date discussion see the paper by Pinto de Moura, Robinson, & Sánchez-Gabites (2010).

I started writing this book while I was a Royal Society University Research Fellow, and many of the results here derive from work done during that time. I am currently supported by an EPSRC Leadership Fellowship, Grant EP/G007470/1. I am extremely grateful to both the Royal Society and to the EPSRC for their support.

I would like to thank Alexandre Carvalho, Peter Friz, Igor Kukavica, José Langa, Eric Olson, Eleonora Pinto de Moura, and Alejandro Vidal López, all of whom have had a hand in material that is presented here. In particular, Eleonora was working on closely-related problems for her doctoral thesis during most of the time that I was writing this book, and our frequent discussions have shaped much of the content and my approach to the material. I had comments on a draft version of the manuscript from Witold Sadowski, Jaime Sánchez-Gabites, and Nicholas Sharples: I am extremely grateful for their helpful and perceptive comments. David Tranah, Clare Dennison, and Emma Walker at Cambridge University Press have been most patient as one deadline after another was missed and extended; that one was finally met (nearly) is due in large part to a kind invitation from Marco Sammartino to Palermo, where I gave a series of lectures on some of the material in this book in November 2009.

Many thanks to my parents and to my mother-in-law; in addition to all their other support, their many days with the children have made this work possible. Finally, of course, thanks to Tania, my wife, and our children Joseph and Kate, who make it all worthwhile; this book is dedicated to them.