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978-0-521-89805-8 - Dimensions, Embeddings, and Attractors
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Introduction

Part I of this book treats four different definitions of dimension, and investigates what being 'finite dimensional' implies in terms of embeddings into Euclidean spaces for each of these definitions.

Whitney (1936) showed that any abstract n-dimensional C^r manifold is C^r-homeomorphic to an analytic submanifold in $\mathbb{R}^{2n+1}$. This book treats embeddings for much more general sets that need not have such a smooth structure; one might say 'fractals', but we will not be concerned with the fractal nature of these sets (whatever one takes that to mean).

We will consider four major definitions of dimension:

(i) The (Lebesgue) covering dimension $\dim(X)$, based on the maximum number of simultaneously intersecting sets in refinements of open covers of X (Chapter 1). This definition is topologically invariant, and is primarily used in the classical and abstract 'Dimension Theory', elegantly developed in Hurewicz & Wallman's 1941 text, and subsequently by Engelking (1978), who updates and extends their treatment.

(ii) The Hausdorff dimension $d_{\mathrm{H}}(X)$, the value of d where the 'd-dimensional Hausdorff measure' of X switches from ∞ to zero (Chapter 2). Hausdorff measures (and hence the Hausdorff dimension) play a large role in geometric measure theory (Federer, 1969), and in the theory of dynamical systems (see Pesin (1997)); the standard reference is Falconer's 1985 tract, and subsequent volumes (Falconer, 1990, 1997).

(iii) The (upper) box-counting dimension $d_{\mathrm{B}}(X)$, essentially the scaling as $\epsilon \to 0$ of $N(X, \epsilon)$, the number of ϵ-balls required to cover X, i.e. $N(X, \epsilon) \sim \epsilon^{-d_{\mathrm{B}}(X)}$ (Chapter 3). This dimension has mainly found application in the field of dynamical systems, see for example Falconer (1990), Eden *et al.* (1994), C. Robinson (1995), and Robinson (2001).

(iv) The Assouad dimension $d_{\mathrm{A}}(X)$, a 'uniform localised' version of the box-counting dimension: if $B(x, \rho)$ denotes the ball of radius ρ centred at $x \in X$, then $N(X \cap B(x, \rho), r) \sim (\rho/r)^{d_{\mathrm{A}}(X)}$ for every $x \in X$ and every $0 < r < \rho$ (Chapter 9). This definition appears unfamiliar outside the area of metric spaces and most results are confined to research papers (e.g. Assouad (1983), Luukkainen (1998), Olson (2002); but see also Heinonen (2001, 2003)).

For any compact metric space (X, ϱ) we will see that

$$\dim(X) \leq d_{\mathrm{H}}(X) \leq d_{\mathrm{B}}(X) \leq d_{\mathrm{A}}(X),$$

and there are examples showing that each of these inequalities can be strict. We will check that each definition satisfies the natural properties of a dimension: monotonicity ($X \subseteq Y$ implies that $d(X) \leq d(Y)$); stability under finite unions ($d(X \cup Y) = \max(d(X), d(Y))$); and the dimension of $\mathbb{R}^n$ is n (a consistent way to interpret this so that it makes sense for all the definitions above is that $d(K) = n$ if K is a compact subset of $\mathbb{R}^n$ that contains an open set). We will also consider how each definition behaves for product sets.

Our main concern will be with the embedding results that are available for each class of 'finite-dimensional' set. The embedding result for sets with finite covering dimension, due to Menger (1926) and Nöbeling (1931) (given as Theorem 1.12 here), is in a class of its own. The result guarantees that when $\dim(X) \leq d$, a generic set of continuous maps from a compact metric space (X, ϱ) into $\mathbb{R}^{2d+1}$ are embeddings.

The results for sets with finite Hausdorff, upper box-counting, and Assouad dimension are of a different cast. They are expressed in terms of 'prevalence' (a version of 'almost every' that is applicable to subsets of infinite-dimensional spaces, introduced independently by Christensen (1973) and Hunt, Sauer, & Yorke (1992), and the subject of Chapter 5), and treat compact subsets of Hilbert and Banach spaces. Using techniques introduced by Hunt & Kaloshin (1999), we show that a 'prevalent' set of continuous linear maps $L : \mathscr{B} \to \mathbb{R}^k$ provide embeddings of X when $d(X - X) < k$, where

$$X - X = \{x_1 - x_2 : \ x_1, x_2 \in X\}$$

and d is one of the above three dimensions (see Figure 1). Note that if one wishes to show that a linear map provides an embedding, i.e. that $Lx = Ly$ implies that $x = y$, this is equivalent to showing that $Lz = 0$ implies that $z = 0$ for $z \in X - X$. This is why the natural condition for such results is one on the 'difference' set $X - X$; but while $d_{\mathrm{B}}(X - X) \leq 2d_{\mathrm{B}}(X)$, there are examples of

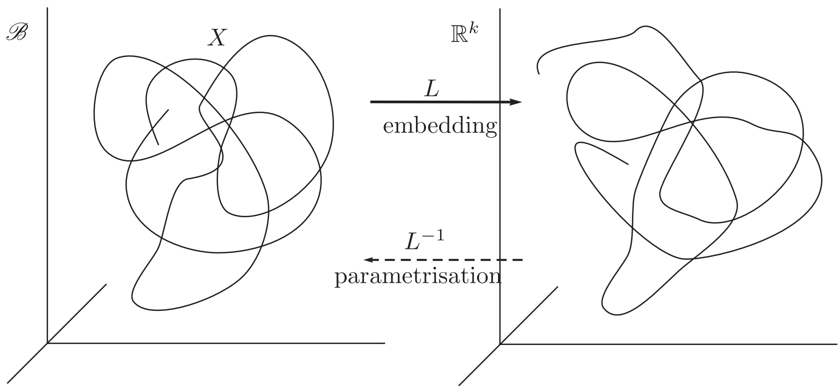

Figure 1 The linear map $L : \mathscr{B} \to \mathbb{R}^k$ embeds X into $\mathbb{R}^k$. The inverse mapping L^{-1} provides a parametrisation of X using k parameters.

sets for which $d_{\mathrm{H}}(X) = 0$ but $d_{\mathrm{H}}(X - X) = \infty$ (and similarly for the Assouad dimension).

Where the embedding results for these three dimensions differ from one another is in the smoothness of the parametrisation of X provided by L^{-1}. In the Hausdorff case this inverse can only be guaranteed to be continuous (Chapter 6); in the upper box-counting case it will be Hölder (Chapter 8); and in the Assouad case it will be Lipschitz to within logarithmic corrections (Chapter 9). Simple examples of orthogonal sequences in ℓ^2 (or related examples in c_0, the space of sequences that tend to zero) show that the results we give cannot be improved when the embedding map L is linear.

Chapter 4 presents an embedding result for subsets X of $\mathbb{R}^N$ with box-counting dimension $d < (N - 1)/2$. The ideas here form the basis of the results for subsets of Hilbert and Banach spaces that follow, and justify the development of the theory of prevalence in Chapter 5 and the definition of various 'thickness exponents' (the thickness exponent itself, the Lipschitz deviation, and the dual thickness) in Chapter 7.

Part II discusses the attractors that arise in certain infinite-dimensional dynamical systems, and the implications of the results of Part I for this class of finite-dimensional sets. In particular, the embedding result for sets with finite box-counting dimension is used toward a proof of an infinite-dimensional version of the Takens time-delay embedding theorem (Chapter 14) and it is shown that a finite-dimensional set of real analytic functions can be parametrised using a finite number of point values (Chapter 15).

Chapter 10 gives a very cursory summary of some elements of the theory of Sobolev spaces and fractional power spaces of linear operators, which are required in order to discuss the applications to partial differential equations. It is shown how the solutions of an abstract semilinear parabolic equation, and of the two-dimensional Navier–Stokes equations, can be used to generate an infinite-dimensional dynamical system whose evolution is described by a nonlinear semigroup.

The global attractor of such a nonlinear semigroup is a compact invariant set that attracts all bounded subsets of the phase space. A sharp condition guaranteeing the existence this global attractor is given in Chapter 11, and it is shown that such an object exists for the semilinear parabolic equation and the Navier–Stokes equations that were treated in the previous chapter.

Chapter 12 provides a method for bounding the upper box-counting dimension of attractors in Banach spaces. While there are powerful techniques available for attractors in Hilbert spaces, these are already presented in a number of other texts, and outlining the more general Banach space technique is more in keeping with the overall approach of this book (the Hilbert space method is covered here in an extended series of exercises). In particular, we show that any attractor of the abstract semilinear parabolic equation introduced in Chapter 10 will be finite-dimensional.

Before proving the final two ‘concrete’ embedding theorems in Chapters 14 and 15, Chapter 13 provides two results that guarantee that an attractor has zero ‘thickness’: we show first that if the attractor consists of smooth functions then its thickness exponent is zero, and then that the attractors of a wide variety of models (which can be written in the abstract semilinear parabolic form) have zero Lipschitz deviation. This, in part, answers a conjecture of Ott, Hunt, & Kaloshin (2006).

Most of the chapters end with a number of exercises. Many of these carry forward portions of the argument that would break the flow of the main text, or discuss related approaches. Full solutions of the exercises are given at the end of the book.

All Hilbert and Banach spaces are real, throughout.

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PART I

Finite-dimensional sets

1

Lebesgue covering dimension

There are a number of definitions of dimension that are invariant under homeomorphisms, i.e. that are topological invariants – in particular, the large and small inductive dimensions, and the Lebesgue covering dimension. Although different a priori, the large inductive dimension and the Lebesgue covering dimension are equal in any metric space (Katětov, 1952; Morita, 1954; Chapter 4 of Engelking, 1978), and all three definitions coincide for separable metric spaces (Proposition III.5 A and Theorem V.8 in Hurewicz & Wallman (1941)). A beautiful exposition of the theory of 'topological dimension' is given in the classic text by Hurewicz & Wallman (1941), which treats separable spaces throughout and makes much capital out of the equivalence of these definitions. Chapter 1 of Engelking (1978) recapitulates these results, while the rest of his book discusses dimension theory in more general spaces in some detail.

This chapter concentrates on one of these definitions, the Lebesgue covering dimension, which we will denote by $\dim(X)$, and refer to simply as the covering dimension. Among the three definitions mentioned above, it is the covering dimension that is most suitable for proving an embedding result: we will show in Theorem 1.12, the central result of this chapter, that if $\dim(X) \leq n$ then a generic set of continuous maps from X into $\mathbb{R}^{2n+1}$ are homeomorphisms, i.e. provide an embedding of X into $\mathbb{R}^{2n+1}$.

There is, unsurprisingly, a topological flavour to the arguments involved here, and consequently they are very different from those in the rest of this book. However, any survey of embedding results for finite-dimensional sets would be incomplete without including the 'fundamental' embedding theorem that is available for sets with finite covering dimension.

1.1 Covering dimension

Let (X, ϱ) be a metric space, and A a subset[1] of X. A *covering* of $A \subseteq X$ is a finite collection $\{U_j\}_{j=1}^r$ of open subsets of X such that

$$A \subseteq \bigcup_{j=1}^{r} U_j.$$

The *order* of a covering is the largest integer n such that there are $n+1$ members of the covering that have a nonempty intersection. A covering β is a *refinement* of a covering α if every member of β is contained in some member of α.

Definition 1.1 A set $A \subseteq X$ has $\dim(A) \leq n$ if every covering has a refinement of order $\leq n$. A set A has $\dim(A) = n$ if $\dim(A) \leq n$ but it is not true that $\dim(A) \leq n-1$.

Clearly dim is a topological invariant. We now prove some elementary properties of the covering dimension, following Munkres (2000) and Edgar (2008).

Proposition 1.2 *Let $B \subseteq A \subseteq X$, with B closed. If* $\dim(A) = n$ *then* $\dim(B) \leq n$.

Proof Let α be a covering of B by open subsets $\{U_j\}$ of X. Cover A by the sets $\{U_j\}$, along with the open set $X \setminus B$. Let β be a refinement of this covering that has order at most n. Then the collection

$$\beta' := \{U \in \beta : U \cap B \neq \emptyset\}$$

is a refinement of α that covers B and has order at most n. □

The assumption that B is closed makes the proof significantly simpler, but the result remains true for an arbitrary subset of A, see Theorem 3.2.13 in Edgar (2008), or Theorem III.1 in Hurewicz & Wallman (1941). However, the following 'sum theorem' is not true unless one of the spaces is closed: in fact, $\dim(X) = n$ if and only if X can be written as the union of $n+1$ subsets all of which have dimension zero (see Theorem III.3 in Hurewicz & Wallman (1941)).

[1] In the context of metric spaces it is somewhat artificial to make the definition in this form, since (A, ϱ) is a metric space in its own right. But our main focus in what follows will be on subsets of Hilbert and Banach spaces, where the underlying linear structure of the ambient space will be significant.

Proposition 1.3 *Let* $X = X_1 \cup X_2$*, where* X_1 *and* X_2 *are closed subspaces of* X *with* $\dim(X_1) \leq n$ *and* $\dim(X_2) \leq n$*. Then* $\dim(X) \leq n$*.*

Of course, it follows that if $X = X_1 \cup \cdots \cup X_k$, each X_j is closed and $\dim(X_j) \leq n$ for every $j = 1, \ldots, k$ then $\dim(X) \leq n$. In fact one can extend this to countable unions of closed sets, see Theorem III.2 in Hurewicz & Wallman (1941) (and Theorem 3.2.11 in Edgar (2008) for the case $n = 1$).

Proof We will say that an open covering α of X has order at most n at points of Y if every point in Y lies in no more than $n + 1$ elements of α.

First we show that any open covering α of X has a refinement that has order at most n at points of X_1. Any such covering of X provides a covering of X_1, which has a refinement β' that has order at most n. For every $V \in \beta'$, there exists an element $U_V \in \alpha$ such that $V \subset U_V$. Then

$$\beta = \{U_V : V \in \beta'\} \cup \{U \setminus X_1 : U \in \alpha\}$$

is the required refinement of α. We can repeat this argument starting with the covering β of X, and obtain a covering γ that refines β and has order at most n at points of X_2.

We now define a further covering of X, which will turn out to be a refinement of α of order at most n. As a first step in our construction, define a map $f : \gamma \to \beta$ by choosing, for each $G \in \gamma$, an $f(G) \in \beta$ such that $G \subset f(G)$ (this is possible since γ refines β). Now for each $B \in \beta$, let

$$d(B) = \{G \in \gamma : f(G) = B\},$$

and let δ be the union of all the sets $d(B)$ (over $B \in \beta$).

Now, δ is a refinement of α, since $d(B) \subset B$ for every $B \in \beta$, and β is a refinement of α. Also, δ still covers X since γ covers X and every $G \in \gamma$ is contained in some $B \in \beta$ (as γ refines β). All that remains is to show that δ has order at most n.

Suppose that $x \in X$ with $x \in d(B_1) \cap \cdots \cap d(B_k)$, with all the $d(B_k)$ distinct (thus $B_1, \ldots, B_k$ are distinct). It follows that for each $j = 1, \ldots, k$, $x \in G_j$ where $f(G_j) = B_j$; since $B_1, \ldots, B_k$ are distinct, so are $G_1, \ldots, G_k$. Thus

$$x \in G_1 \cap \cdots \cap G_k \subset d(B_1) \cap \cdots \cap d(B_k) \subset B_1 \cap \cdots \cap B_k.$$

If $x \in X_1$ then $k \leq n + 1$ because β has order at most n at points of X_1; and if $x \in X_2$ then $k \leq n + 1$ because γ has order at most n at points of X_2. □

We do not prove a result on the covering dimension of products here, although it is the case that $\dim(X \times Y) \leq \dim(X) + \dim(Y)$ (Theorem III.4 in

Hurewicz & Wallman (1941)): this can be proved as a corollary of a characterisation of the covering dimension in terms of the upper box-counting dimension, see Exercise 3.4.

1.2 The covering dimension of I_n

It is by no means trivial to show that the covering dimension of $\mathbb{R}^n$ is n. Note that it suffices to show that $\dim(I_n) = n$, where $I_n = [-\frac{1}{2}, \frac{1}{2}]^n$ denotes the unit cube in $\mathbb{R}^n$, since as remarked after Proposition 1.3, the covering dimension is in fact stable under countable unions of closed sets.

We refer to Theorem 50.6 in Munkres (2000) for a direct proof of the upper bound on $\dim(I_n)$ (see also Exercise 1.2 for compact subsets of $\mathbb{R}^2$). One can also deduce the upper bound from the general fact that the covering dimension is bounded by the Hausdorff dimension (Theorem 2.11); it is very simple to show that the Hausdorff dimension of a subset of $\mathbb{R}^n$ is bounded by n (Proposition 2.8(iii)).

While the proof of the upper bound is more notationally awkward than technically difficult, the proof of the lower bound involves the powerful Brouwer Fixed Point Theorem (see IV (C) of Hurewicz & Wallman (1941) for a proof).

Theorem 1.4 *Any continuous map $f : I_n \to I_n$ has a fixed point, i.e. there exists an $x_0 \in I_n$ such that $f(x_0) = x_0$.*

We give a proof of the lower bound (essentially the 'Lebesgue Covering Theorem') adapted from Hurewicz & Wallman's book, for the two-dimensional unit cube $I_2 = [-\frac{1}{2}, \frac{1}{2}]^2$. The general result (for I_n) is not significantly more involved, but the argument can be somewhat simplified in this case without losing its essential flavour. (An alternative proof of a similar two-dimensional result is given as Theorem 3.3.4 in Edgar (2008).) Before the proof we introduce some notation.

Given a set $U \subset (X, \varrho)$ we define the diameter of U, written $|U|$, as

$$|U| = \operatorname{diam}(U) = \sup_{u_1, u_2 \in U} \varrho(u_1, u_2).$$

(We only use the notation $\operatorname{diam}(U)$ when $|U|$ would be ambiguous.) The *mesh size* of a covering of A is the largest of the diameters of the elements of the covering.

For two sets $A, B \subset X$ we write

$$\operatorname{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} \varrho(a, b)$$

for the Hausdorff semidistance between A and B. Note that if B is closed then $\text{dist}(A, B) = 0$ implies that $A \subseteq B$.

Theorem 1.5 *Let* $I_2 = [-\frac{1}{2}, \frac{1}{2}]^2 \subset \mathbb{R}^2$. *Then* $\dim(I_2) \geq 2$.

Proof We want to show that any covering α of I_2 with sufficiently small mesh size contains at least three sets with nonempty intersection. To this end, take a covering α with mesh size < 1 so that no element of the covering contains points of opposite faces.

The first step is to construct a refinement $\tilde{\alpha}$ of α consisting of closed, rather than open, sets. To do this, observe that every $x \in I_2$ is contained in some $U_x \in \alpha$, and we can find an open set V_x such that $x \in V_x \subset \bar{V}_x \subset U_x$. Since I_2 is compact and $\{V_x : x \in I_2\}$ is an open cover of I_2, there is a finite subcover $\{V_{x_j}\}$. We take $\tilde{\alpha}$ to be the collection of all the closed sets $\{\bar{V}_{x_j}\}$. By construction this is a refinement of α consisting of closed sets.

We now show that $\tilde{\alpha}$ contains at least three sets with nonempty intersection, from which it is immediate (since $\tilde{\alpha}$ is a refinement of α) that α contains at least three sets with nonempty intersection.

Let Γ_1 denote the side of I_2 with $x = -\frac{1}{2}$, Γ_1' the side with $x = \frac{1}{2}$, Γ_2 the side with $y = -\frac{1}{2}$, and Γ_2' the side with $y = \frac{1}{2}$. Let L_1 denote the union of those elements of $\tilde{\alpha}$ that intersect Γ_1; L_2 the union of those elements of $\tilde{\alpha}$ that are not in L_1 and intersect Γ_2; and let L_3 be the union of all the other elements of $\tilde{\alpha}$ (those that intersect neither Γ_1 nor Γ_2). See Figure 1.1(a).

If we define $K_1 = L_1 \cap L_3$ then K_1 separates Γ_1 and Γ_1' in I_2, i.e. there exist open sets U_1 and U_1' such

$$I_2 \setminus K_1 = U_1 \cup U_1', \qquad U_1 \cap U_1' = \emptyset$$

and $\Gamma_1 \subset U_1$, $\Gamma_1' \subset U_1'$. The set $K_2' = L_1 \cap L_2 \cap L_3$ separates $\Gamma_2 \cap K_1$ from $\Gamma_2' \cap K_1$ in K_1. One can then find a new closed set K_2, with $K_2 \cap K_1 \subseteq K_2'$, that separates Γ_2 and Γ_2' in I_2, i.e. such that there exist open sets U_2 and U_2' such that

$$I_2 \setminus K_2 = U_2 \cup U_2', \qquad U_2 \cap U_2' = \emptyset$$

and $\Gamma_2 \subset U_2$, $\Gamma_2' \subset U_2'$. These constructions are illustrated in Figure 1.1(b). (If the 'proof by diagram' of this last step is unconvincing, see IV.3 A) in Hurewicz & Wallman (1941), or Exercise 1.3.)

Now for each $x \in I_2$, let $v(x)$ be the 2-vector with components

$$v_i(x) = \begin{cases} \text{dist}(x, K_i) & x \in U_i, \\ 0 & x \in K_i, \\ -\text{dist}(x, K_i) & x \in U_i', \end{cases}$$