Part I

Introduction

Cambridge University Press
978-0-521-89772-3 - Statistical Signal Processing of Complex-Valued Data: The Theory of Improper and Noncircular Signals
Peter J. Schreier and Louis L. Scharf
Excerpt
More information

1 The origins and uses of complex signals

Engineering and applied science rely heavily on complex variables and complex analysis to model and analyze real physical effects. Why should this be so? That is, why should *real* measurable effects be represented by *complex* signals? The ready answer is that one complex signal (or channel) can carry information about two real signals (or two real channels), and the algebra and geometry of analyzing these two real signals as if they were one complex signal brings economies and insights that would not otherwise emerge. But ready answers beg for clarity. In this chapter we aim to provide it. In the bargain, we intend to clarify the language of engineers and applied scientists who casually speak of complex velocities, complex electromagnetic fields, complex baseband signals, complex channels, and so on, when what they are really speaking of is the x- and y-coordinates of velocity, the x- and y-components of an electric field, the in-phase and quadrature components of a modulating waveform, and the sine and cosine channels of a modulator or demodulator.

For electromagnetics, oceanography, atmospheric science, and other disciplines where two-dimensional trajectories bring insight into the underlying physics, it is the complex representation of an ellipse that motivates an interest in complex analysis. For communication theory and signal processing, where amplitude and phase modulations carry information, it is the complex baseband representation of a real bandpass signal that motivates an interest in complex analysis.

In Section 1.1, we shall begin with an elementary introduction to complex representations for Cartesian coordinates and two-dimensional signals. Then we shall proceed to a discussion of phasors and Lissajous figures in Sections 1.2 and 1.3. We will find that phasors are a complex representation for the motion of an undamped harmonic oscillator and Lissajous figures are a complex representation for polarized electromagnetic fields. The study of communication signals in Section 1.4 then leads to the Hilbert transform, the complex analytic signal, and various principles for modulating signals. Section 1.5 demonstrates how real signals can be loaded into the real and imaginary parts of a complex signal in order to make efficient use of the fast Fourier transform (FFT).

The second half of this chapter deals with complex *random* variables and signals. In Section 1.6, we introduce the univariate complex Gaussian probability density function (pdf) as an alternative parameterization for the bivariate pdf of two real correlated Gaussian random variables. We will see that the well-known form of the univariate complex Gaussian pdf models only a special case of the bivariate real pdf, where the two real random variables are independent and have equal variances. This special case is called *proper* or *circular*, and it corresponds to a uniform phase distribution of the complex random variable. In general, however, the complex Gaussian pdf depends not only on the variance but also on another term, which we will call the *complementary variance*. In Section 1.7, we extend this discussion to complex random signals. Using the polarization ellipse as an example, we will find an interplay of reality/complexity, propriety/impropriety, and wide-sense stationarity/nonstationarity. Section 1.8 provides a first glance at the mathematical framework that underpins the study of complex random variables in this book. Finally, Section 1.9 gives a brief survey of some recent papers that apply the theory of improper and noncircular complex random signals in communications, array processing, machine learning, acoustics, optics, and oceanography.

1.1 Cartesian, polar, and complex representations of two-dimensional signals

It is commonplace to represent two Cartesian coordinates (u, v) in their two polar coordinates (A, θ), or as the single complex coordinate $x = u + \mathrm{j}v = A\mathrm{e}^{\mathrm{j}\theta}$. The real coordinates $(u, v) \longleftrightarrow (A, \theta)$ are thus equivalent to the complex coordinates $u + \mathrm{j}v \longleftrightarrow A\mathrm{e}^{\mathrm{j}\theta}$. The virtue of this complex representation is that it leads to an economical algebra and an evocative geometry, especially when polar coordinates A and θ are used. This virtue extends to vector-valued coordinates $(\mathbf{u}, \mathbf{v})$, with complex representation $\mathbf{x} = \mathbf{u} + \mathrm{j}\mathbf{v}$. For example, $\mathbf{x}$ could be a mega-vector composed by stacking scan lines from a stereoscopic image, in which case $\mathbf{u}$ would be the image recorded by camera one and $\mathbf{v}$ would be the image recorded by camera two. In oceanographic applications, $\mathbf{u}$ and $\mathbf{v}$ could be the two orthogonal components of surface velocity and $\mathbf{x}$ would be the complex velocity. Or $\mathbf{x}$ could be a window's worth of a discrete-time communications signal. In the context of communications, radar, and sonar, $\mathbf{u}$ and $\mathbf{v}$ are called the *in-phase* and *quadrature* components, respectively, and they are obtained as sampled-data versions of a continuous-time signal that has been demodulated with a quadrature demodulator. The quadrature demodulator itself is designed to extract a baseband information-bearing signal from a passband carrying signal. This is explained in more detail in Section 1.4.

The virtue of complex representations extends to the analysis of time-varying coordinates $(u(t), v(t))$, which we call two-dimensional signals, and which we represent as the complex signal $x(t) = u(t) + \mathrm{j}v(t) = A(t)\mathrm{e}^{\mathrm{j}\theta(t)}$. Of course, the next generalization of this narrative would be to vector-valued complex signals $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^{\mathrm{T}}$, a generalization that produces technical difficulties, but not conceptual ones. The two best examples are complex-demodulated signals in a multi-sensor antenna array, in which case $x_k(t)$ is the complex signal recorded at sensor k, and complex-demodulated signals in spectral subbands of a wideband communication signal, in which case $x_k(t)$ is the complex signal recorded in subband k. When these signals are themselves sampled in time, then the vector-valued discrete-time sequence is $\mathbf{x}[n]$, with $\mathbf{x}[n] = \mathbf{x}(nT)$ a sampled-data version of $\mathbf{x}(t)$.

This introductory account of complex signals gives us the chance to remake a very important point. In engineering and applied science, measured signals are *real*. Correspondingly, in all of our examples, the components u and v are *real*. It is only our representation x that is complex. Thus one channel's worth of complex signal serves to represent two channels' worth of real signals. There is no fundamental reason why this would have to be done. We aim to make the point in this book that the algebraic economies, probabilistic computations, and geometrical insights that accrue to complex representations justify their use. The examples of the next several sections give a preview of the power of complex representations.

1.2 Simple harmonic oscillator and phasors

The damped harmonic oscillator models damped pendulums and second-order electrical and mechanical systems. A measurement (of position or voltage) in such a system obeys the second-order, homogeneous, linear differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(t) + 2\xi\omega_0 \frac{\mathrm{d}}{\mathrm{d}t}u(t) + \omega_0^2 u(t) = 0. \tag{1.1}$$

The corresponding characteristic equation is

$$s^2 + 2\xi\omega_0 s + \omega_0^2 = 0. \tag{1.2}$$

If the damping coefficient ξ satisfies $0 \leq \xi < 1$, the system is called *underdamped*, and the quadratic equation (1.2) has two complex conjugate roots $s_1 = -\xi\omega_0 + \mathrm{j}\sqrt{1-\xi^2}\omega_0$ and $s_2 = s_1^*$. The real homogeneous response of the damped harmonic oscillator is then

$$u(t) = A\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{s_1 t} + A\mathrm{e}^{-\mathrm{j}\theta}e^{s_1^* t} = \mathrm{Re}\,\{A\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{s_1 t}\} = A\mathrm{e}^{-\xi\omega_0 t}\cos(\sqrt{1-\xi^2}\omega_0 t + \theta), \tag{1.3}$$

and A and θ may be determined from the initial values of $u(t)$ and $(\mathrm{d}/\mathrm{d}t)u(t)$ at $t = 0$. The real response (1.3) is the sum of two complex modal responses, or the real part of one of them. In anticipation of our continuing development, we might say that $A\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{s_1 t}$ is a complex representation of the real signal $u(t)$.

For the *undamped* system with damping coefficient $\xi = 0$, we have $s_1 = \mathrm{j}\omega_0$ and the solution is

$$u(t) = \mathrm{Re}\,\{A\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{\mathrm{j}\omega_0 t}\} = A\cos(\omega_0 t + \theta). \tag{1.4}$$

In this case, $A\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{\mathrm{j}\omega_0 t}$ is the complex representation of the real signal $A\cos(\omega_0 t + \theta)$. The complex signal in its polar form

$$x(t) = A\mathrm{e}^{\mathrm{j}(\omega_0 t + \theta)} = A\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{\mathrm{j}\omega_0 t}, \qquad t \in \mathbb{R}, \tag{1.5}$$

is called a *rotating phasor*. The *rotator* $\mathrm{e}^{\mathrm{j}\omega_0 t}$ rotates the *stationary phasor* $A\mathrm{e}^{\mathrm{j}\theta}$ at the angular rate of ω_0 radians per second. The rotating phasor is periodic with period $2\pi/\omega_0$, thus overwriting itself every $2\pi/\omega_0$ seconds. Euler's identity allows us to express the rotating phasor in its Cartesian form as

$$x(t) = A\cos(\omega_0 t + \theta) + \mathrm{j}A\sin(\omega_0 t + \theta). \tag{1.6}$$

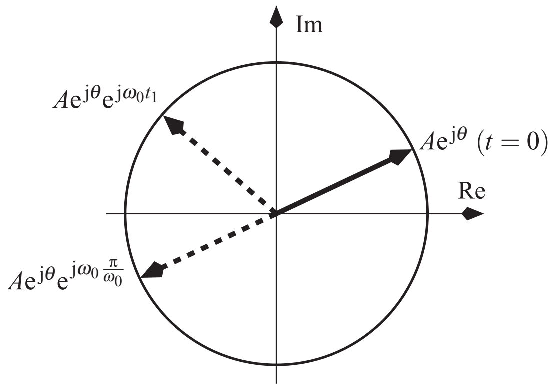

Figure 1.1 Stationary and rotating phasors.

Thus, the complex representation of the undamped simple harmonic oscillator turns out to be the trajectory in the complex plane of a rotating phasor of radian frequency ω_0, with starting point $Ae^{j\theta}$ at $t = 0$. The rotating phasor of Fig. 1.1 is illustrative.

The rotating phasor is one of the most fundamental complex signals we shall encounter in this book, as it is a basic building block for more complicated signals. As we build these more complicated signals, we will allow A and θ to be correlated random processes.

1.3 Lissajous figures, ellipses, and electromagnetic polarization

We might say that the circularly rotating phasor $x(t) = Ae^{j\theta}e^{j\omega_0 t} = A\cos(\omega_0 t + \theta) + jA\sin(\omega_0 t + \theta)$ is the simplest of Lissajous figures, consisting of real and imaginary parts that are $\pi/2$ radians out of phase. A more general Lissajous figure allows complex signals of the form

$$x(t) = u(t) + jv(t) = A_u\cos(\omega_0 t + \theta_u) + jA_v\cos(\omega_0 t + \theta_v). \tag{1.7}$$

Here the real part $u(t)$ and the imaginary part $v(t)$ can be mismatched in amplitude and phase. This Lissajous figure overwrites itself with period $2\pi/\omega_0$ and turns out an ellipse in the complex plane. (This is still not the most general Lissajous figure, since Lissajous figures generally also allow different frequencies in the u- and v-components.)

In electromagnetic theory, this complex signal would be the time-varying position of the electric field vector in the (u, v)-plane perpendicular to the direction of propagation. Over time, as the electric field vector propagates, it turns out an elliptical corkscrew in three-dimensional space. But in the two-dimensional plane perpendicular to the direction of propagation, it turns out an ellipse, so the electric field is said to be elliptically polarized. As this representation shows, the elliptical polarization may be modeled, and in fact produced, by the superposition of a one-dimensional, *linearly polarized*,

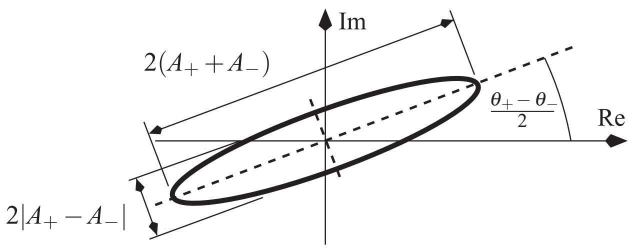

Figure 1.2 A typical polarization ellipse.

component of the form $A_u \cos(\omega_0 t + \theta_u)$ in the u-direction and another of the form $A_v \sin(\omega_0 t + \theta_v)$ in the v-direction.

But there is more. Euler's identity may be used to write the electric field vector as

$$\begin{aligned} x(t) &= \tfrac{1}{2} A_u e^{j\theta_u} e^{j\omega_0 t} + \tfrac{1}{2} A_u e^{-j\theta_u} e^{-j\omega_0 t} + \tfrac{1}{2} j A_v e^{j\theta_v} e^{j\omega_0 t} + \tfrac{1}{2} j A_v e^{-j\theta_v} e^{-j\omega_0 t} \\ &= \underbrace{\tfrac{1}{2}\left(A_u e^{j\theta_u} + jA_v e^{j\theta_v}\right)}_{A_+ e^{j\theta_+}} e^{j\omega_0 t} + \underbrace{\tfrac{1}{2}\left(A_u e^{-j\theta_u} + jA_v e^{-j\theta_v}\right)}_{A_- e^{-j\theta_-}} e^{-j\omega_0 t}. \end{aligned} \tag{1.8}$$

This representation of the two-dimensional electric field shows it to be the superposition of a two-dimensional, circularly polarized, component of the form $A_+ e^{j\theta_+} e^{j\omega_0 t}$ and another of the form $A_- e^{-j\theta_-} e^{-j\omega_0 t}$. The first rotates *counterclockwise* (CCW) and is said to be *left-circularly polarized*. The second rotates *clockwise* (CW) and is said to be *right-circularly polarized*. In this representation, the complex constants $A_+ e^{j\theta_+}$ and $A_- e^{-j\theta_-}$ fix the amplitude and phase of their respective circularly polarized components.

The circular representation of the ellipse makes it easy to determine the orientation of the ellipse and the lengths of the major and minor axes. In fact, by noting that the magnitude-squared of $x(t)$ is $|x(t)|^2 = A_+^2 + 2A_+A_- \cos(\theta_+ + \theta_- + 2\omega_0 t) + A_-^2$, it is easy to see that $|x(t)|^2$ has a maximum value of $(A_+ + A_-)^2$ at $\theta_+ + \theta_- + 2\omega_0 t = 2k\pi$, and a minimum value of $(A_+ - A_-)^2$ at $\theta_+ + \theta_- + 2\omega_0 t = (2k+1)\pi$. This orients the major axis of the ellipse at angle $(\theta_+ - \theta_-)/2$ and fixes the major and minor axis lengths at $2(A_+ + A_-)$ and $2|A_+ - A_-|$. A typical polarization ellipse is illustrated in Fig. 1.2.

Jones calculus

It is clear that the polarization ellipse $x(t)$ may be parameterized either by four real parameters $(A_u, A_v, \theta_u, \theta_v)$ or by two complex parameters $(A_+ e^{j\theta_+}, A_- e^{-j\theta_-})$. In the first case, we modulate the real basis $(\cos(\omega_0 t), \sin(\omega_0 t))$, and in the second case, we modulate the complex basis $(e^{j\omega_0 t}, e^{-j\omega_0 t})$. If we are interested only in the path that the electric field vector describes, and do not need to evaluate $x(t_0)$ at a particular time t_0, knowing the phase *differences* $\theta_u - \theta_v$ or $\theta_+ - \theta_-$ rather than the phases themselves is sufficient. The choice of parameterization – whether real or complex – is somewhat arbitrary, but it is common to use the *Jones vector* $[A_u, A_v e^{j(\theta_u - \theta_v)}]$ to describe the state of polarization. This is illustrated in the following example.

Example 1.1. The Jones vectors for four basic states of polarization are (note that we do not follow the convention of normalizing Jones vectors to unit norm):

$$\begin{bmatrix}1\\0\end{bmatrix} \longleftrightarrow x(t) = \cos(\omega_0 t) \qquad \text{horizontal, linear polarization,}$$

$$\begin{bmatrix}0\\1\end{bmatrix} \longleftrightarrow x(t) = \mathrm{j}\cos(\omega_0 t) \qquad \text{vertical, linear polarization,}$$

$$\begin{bmatrix}1\\j\end{bmatrix} \longleftrightarrow x(t) = \mathrm{e}^{\mathrm{j}\omega_0 t} \qquad \text{CCW (left-) circular polarization,}$$

$$\begin{bmatrix}1\\-j\end{bmatrix} \longleftrightarrow x(t) = \mathrm{e}^{-\mathrm{j}\omega_0 t} \qquad \text{CW (right-) circular polarization.}$$

Various polarization filters can be coded with two-by-two complex matrices that selectively pass components of the polarization. For example, consider these two polarization filters, and their corresponding Jones matrices:

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \qquad \text{horizontal, linear polarizer,}$$

$$\frac{1}{2}\begin{bmatrix}1 & -\mathrm{j}\\ \mathrm{j} & 1\end{bmatrix} \qquad \text{CCW (left-)circular polarizer.}$$

The first of these passes horizontal linear polarization and rejects vertical linear polarization. Such polarizers are used to reduce vertically polarized glare in Polaroid sunglasses. The second passes CCW circular polarization and rejects CW circular polarization. And so on.

1.4 Complex modulation, the Hilbert transform, and complex analytic signals

When analyzing the damped harmonic oscillator or the elliptically polarized electric field, the appropriate complex representations present themselves naturally. We now establish that this is so, as well, in the theory of modulation. Here the game is to modulate a *baseband*, information-bearing, signal onto a *passband* carrier signal that can be radiated from a real antenna onto a real channel. When the aim is to transmit information from here to there, then the channel may be "air," cable, or fiber. When the aim is to transmit information from now to then, then the channel may be a magnetic recording channel.

Actually, since a sinusoidal carrier signal can be modulated in amplitude and phase, the game is to modulate *two* information-bearing signals onto a carrier, suggesting again that *one complex* signal might serve to represent these two real signals and provide insight into how they should be designed. In fact, as we shall see, without the notion of a *complex analytic signal*, electrical engineers might never have discovered the Hilbert transform and single-sideband (SSB) modulation as the most spectrally efficient way to modulate *one* real channel of baseband information onto a passband carrier. Thus modulation theory provides the proper context for the study of the Hilbert transform and complex analytic signals.

1.4.1 Complex modulation using the complex envelope

Let us begin with two *real* information-bearing signals $u(t)$ and $v(t)$, which are combined in a *complex baseband signal* as

$$\begin{aligned} x(t) &= u(t) + \mathrm{j}v(t) \\ &= A(t)\mathrm{e}^{\mathrm{j}\theta(t)} = A(t)\cos\theta(t) + \mathrm{j}A(t)\sin\theta(t). \end{aligned} \tag{1.9}$$

The amplitude $A(t)$ and phase $\theta(t)$ are real. We take $u(t)$ and $v(t)$ to be lowpass signals with Fourier transforms supported on a baseband interval of $-\Omega < \omega < \Omega$. The representation $A(t)\mathrm{e}^{\mathrm{j}\theta(t)}$ is a generalization of the stationary phasor, wherein the fixed radius and angle of a phasor are replaced by a time-varying radius and angle. It is a simple matter to go back and forth between $x(t)$ and $(u(t), v(t))$ and $(A(t), \theta(t))$.

From $x(t)$ we propose to construct the *real* passband signal

$$p(t) = \mathrm{Re}\,\{x(t)\mathrm{e}^{\mathrm{j}\omega_0 t}\} = A(t)\cos(\omega_0 t + \theta(t)) = u(t)\cos(\omega_0 t) - v(t)\sin(\omega_0 t). \tag{1.10}$$

In accordance with standard communications terminology, we call $x(t)$ the complex baseband signal or *complex envelope* of $p(t)$, $A(t)$ and $\theta(t)$ the amplitude and phase of the complex envelope, and $u(t)$ and $v(t)$ the *in-phase* and *quadrature(-phase) components*. The term "quadrature component" refers to the fact that it is in phase quadrature ($+\pi/2$ out of phase) with respect to the in-phase component.

We say the complex envelope $x(t)$ complex-modulates the *complex carrier* $\mathrm{e}^{\mathrm{j}\omega_0 t}$, when what we really mean is that the *real* amplitude and phase $(A(t), \theta(t))$ real-modulate the amplitude and phase of the real carrier $\cos(\omega_0 t)$; or the in-phase and quadrature signals $(u(t), v(t))$ real-modulate the *real* in-phase carrier $\cos(\omega_0 t)$ and the *real* quadrature carrier $\sin(\omega_0 t)$. These are three equivalent ways of saying exactly the same thing. Figure 1.3(a) suggests a diagram for complex modulation. In Fig. 1.3(b), we stress the point that complex channels are actually two parallel real channels.

It is worth noting that when $\theta(t)$ is constant (say zero), then modulation is amplitude modulation only. In the complex plane, the complex baseband signal $x(t)$ writes out a trajectory $x(t) = A(t)$ that does not leave the real line. When $A(t)$ is constant (say 1), then modulation is phase modulation only. In the complex plane, the complex baseband signal $x(t)$ writes out a trajectory $x(t) = \mathrm{e}^{\mathrm{j}\theta(t)}$ that does not leave the unit circle. In general quadrature modulation, both $A(t)$ and $\theta(t)$ are time-varying, and they combine to write out quite arbitrary trajectories in the complex plane. These trajectories are composed of real part $u(t) = A(t)\cos\theta(t)$ and imaginary part $v(t) = A(t)\sin\theta(t)$, or of amplitude $A(t)$ and phase $\theta(t)$.

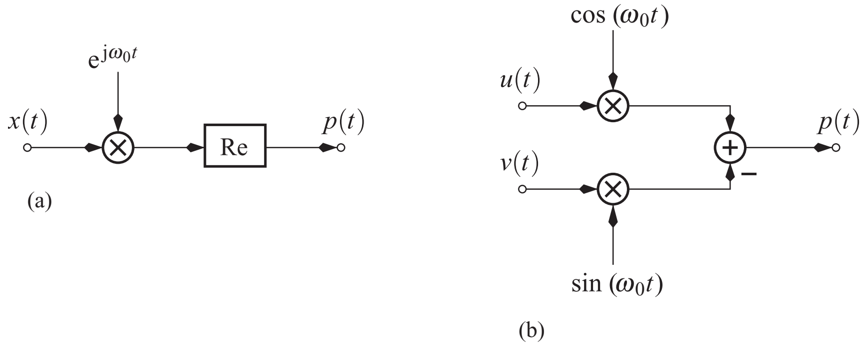

Figure 1.3 (a) Complex and (b) quadrature modulation.

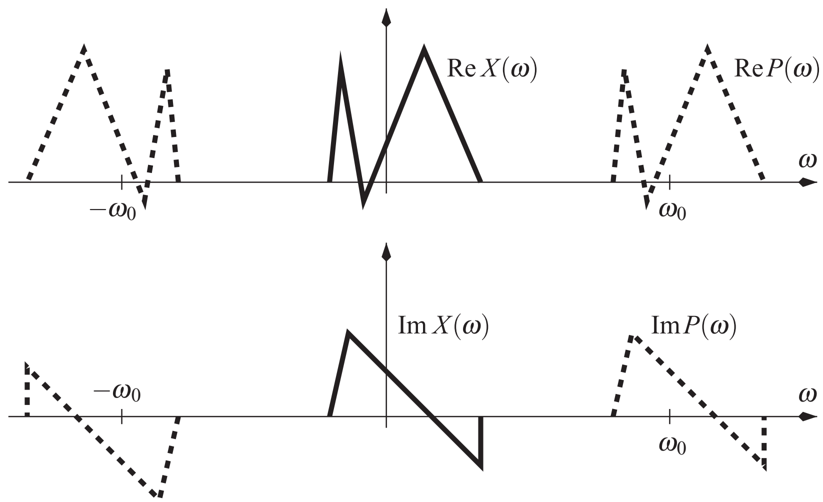

Figure 1.4 Baseband spectrum $X(\omega)$ (solid line) and passband spectrum $P(\omega)$ (dashed line).

If the complex signal $x(t)$ has Fourier transform $X(\omega)$, denoted $x(t) \longleftrightarrow X(\omega)$, then $x(t)e^{j\omega_0 t} \longleftrightarrow X(\omega - \omega_0)$ and $x^*(t) \longleftrightarrow X^*(-\omega)$. Thus,

$$p(t) = \operatorname{Re}\{x(t)e^{j\omega_0 t}\} = \tfrac{1}{2}x(t)e^{j\omega_0 t} + \tfrac{1}{2}x^*(t)e^{-j\omega_0 t} \tag{1.11}$$

has Hermitian-symmetric Fourier transform

$$P(\omega) = \tfrac{1}{2}X(\omega - \omega_0) + \tfrac{1}{2}X^*(-\omega - \omega_0). \tag{1.12}$$

Because $p(t)$ is real its Fourier transform satisfies $P(\omega) = P^*(-\omega)$. Thus, the real part of $P(\omega)$ is even, and the imaginary part is odd. Moreover, the magnitude $|P(\omega)|$ is even, and the phase $\measuredangle P(\omega)$ is odd. Fanciful spectra $X(\omega)$ and $P(\omega)$ are illustrated in Fig. 1.4.

1.4.2 The Hilbert transform, phase splitter, and analytic signal

If the complex baseband signal $x(t)$ can be recovered from the passband signal $p(t)$, then the two real channels $u(t)$ and $v(t)$ can be easily recovered as $u(t) = \text{Re } x(t) = \frac{1}{2}[x(t) + x^*(t)]$ and $v(t) = \text{Im } x(t) = [1/(2\text{j})][x(t) - x^*(t)]$. But how is $x(t)$ to be recovered from $p(t)$?

The *real* operator Re in the definition of $p(t)$ is applied to the complex signal $x(t)\text{e}^{\text{j}\omega_0 t}$ and returns the real signal $p(t)$. Suppose there existed an inverse operator Φ, i.e., a linear, convolutional, *complex* operator, that could be applied to the real signal $p(t)$ and return the complex signal $x(t)\text{e}^{\text{j}\omega_0 t}$. Then this complex signal could be complex-demodulated for $x(t) = \text{e}^{-\text{j}\omega_0 t}\text{e}^{\text{j}\omega_0 t}x(t)$. The complex operator Φ would have to be defined by an impulse response $\phi(t) \longleftrightarrow \Phi(\omega)$, whose Fourier transform $\Phi(\omega)$ were zero for negative frequencies and 2 for positive frequencies, in order to return the signal $x(t)\text{e}^{\text{j}\omega_0 t} \longleftrightarrow X(\omega - \omega_0)$.

This brings us to the Hilbert transform, the phase splitter, and the complex analytic signal. The Hilbert transform of a signal $p(t)$ is denoted $\hat{p}(t)$, and defined as the linear shift-invariant operation

$$\hat{p}(t) = (h * p)(t) \triangleq \int_{-\infty}^{\infty} h(t-\tau)p(\tau)\text{d}\tau \longleftrightarrow (HP)(\omega) \triangleq H(\omega)P(\omega) = \hat{P}(\omega). \tag{1.13}$$

The impulse response $h(t)$ and complex frequency response $H(\omega)$ of the Hilbert transform are defined to be, for $t \in \mathbb{R}$ and $\omega \in \mathbb{R}$,

$$h(t) = \frac{1}{\pi t} \longleftrightarrow -\text{j}\ \text{sgn}(\omega) = H(\omega). \tag{1.14}$$

Here sgn(ω) is the function

$$\text{sgn}(\omega) = \begin{cases} 1, & \omega > 0, \\ 0, & \omega = 0, \\ -1, & \omega < 0. \end{cases} \tag{1.15}$$

So $h(t)$ is real and odd, and $H(\omega)$ is imaginary and odd. From the Hilbert transform $h(t) \longleftrightarrow H(\omega)$ we define the *phase splitter*

$$\phi(t) = \delta(t) + \text{j}h(t) \longleftrightarrow 1 - \text{j}^2\ \text{sgn}(\omega) = 2\Gamma(\omega) = \Phi(\omega). \tag{1.16}$$

The complex frequency response of the phase splitter is $\Phi(\omega) = 2\Gamma(\omega)$, where $\Gamma(\omega)$ is the standard unit-step function. The convolution of the complex filter $\phi(t)$ and the real signal $p(t)$ produces the *analytic signal* $y(t) = p(t) + \text{j}\hat{p}(t)$, with Fourier transform identity

$$y(t) = (\phi * p)(t) = p(t) + \text{j}\hat{p}(t) \longleftrightarrow P(\omega) + \text{sgn}(\omega)P(\omega) = 2(\Gamma P)(\omega) = Y(\omega). \tag{1.17}$$

Recall that the Fourier transform $P(\omega)$ of a real signal $p(t)$ has Hermitian symmetry $P(-\omega) = P^*(\omega)$, so $P(\omega)$ for $\omega < 0$ is redundant. In the polar representation