

Cambridge University Press

978-0-521-89730-3 - Sub-Riemannian Geometry: General Theory and Examples

Ovidiu Calin and Der-Chen Chang

Table of Contents

[More information](#)


---

## Contents

---

Preface *page xi*

### Part I General Theory

<b>1</b>	<b>Introductory Chapter</b>	3
1.1	Differentiable Manifolds	3
1.2	Submanifolds	4
1.3	Distributions	4
1.4	Integral Curves of a Vector Field	5
1.5	Independent One-Forms	9
1.6	Distributions Defined by One-Forms	11
1.7	Integrability of One-Forms	13
1.8	Elliptic Functions	16
1.9	Exterior Differential Systems	17
1.10	Formulas Involving Lie Derivative	22
1.11	Pfaff Systems	24
1.12	Characteristic Vector Fields	26
1.13	Lagrange–Charpit Method	29
1.14	Eiconal Equation on the Euclidean Space	34
1.15	Hamilton–Jacobi Equation on $\mathbb{R}^n$	35
<b>2</b>	<b>Basic Properties</b>	37
2.1	Sub-Riemannian Manifolds	37
2.2	The Existence of Sub-Riemannian Metrics	38
2.3	Systems of Orthonormal Vector Fields at a Point	39
2.4	Bracket-Generating Distributions	41
2.5	Non-Bracket-Generating Distributions	42
2.6	Cyclic Bracket Structures	45
2.7	Strong Bracket-Generating Condition	46
2.8	Nilpotent Distributions	47
2.9	The Horizontal Gradient	49

Cambridge University Press

978-0-521-89730-3 - Sub-Riemannian Geometry: General Theory and Examples

Ovidiu Calin and Der-Chen Chang

Table of Contents

[More information](#)

viii	Contents	
2.10	The Intrinsic and Extrinsic Ideals	56
2.11	The Induced Connection and Curvature Forms	60
2.12	The Iterated Extrinsic Ideals	61
<b>3</b>	<b>Horizontal Connectivity</b>	<b>65</b>
3.1	Teleman's Theorem	65
3.2	Carathéodory's Theorem	73
3.3	Thermodynamical Interpretation	73
3.4	A Global Nonconnectivity Example	75
3.5	Chow's Theorem	78
<b>4</b>	<b>The Hamilton–Jacobi Theory</b>	<b>83</b>
4.1	The Hamilton–Jacobi Equation	83
4.2	Length-Minimizing Horizontal Curves	86
4.3	An Example: The Heisenberg Distribution	89
4.4	Sub-Riemannian Eiconal Equation	92
4.5	Solving the Hamilton–Jacobi Equation	96
<b>5</b>	<b>The Hamiltonian Formalism</b>	<b>98</b>
5.1	The Hamiltonian Function	98
5.2	Normal Geodesics and Their Properties	102
5.3	The Nonholonomic Constraint	106
5.4	The Covariant Sub-Riemannian Metric	108
5.5	Covariant and Contravariant Sub-Riemannian Metrics	110
5.6	The Acceleration Along a Horizontal Curve	113
5.7	Horizontal and Cartesian Components	113
5.8	Normal Geodesics as Length-Minimizing Curves	114
5.9	Eigenvectors of the Contravariant Metric	116
5.10	Poisson Formalism	118
5.11	Invariants of a Distribution	121
<b>6</b>	<b>Lagrangian Formalism</b>	<b>124</b>
6.1	Lagrange Multipliers	124
6.2	Singular Minimizers	128
6.3	Regular Implies Normal	130
6.4	The Euler–Lagrange Equations	132
<b>7</b>	<b>Connections on Sub-Riemannian Manifolds</b>	<b>137</b>
7.1	The Horizontal Connection	137
7.2	The Torsion of the Horizontal Connection	141
7.3	Horizontal Divergence	142
7.4	Connections on Sub-Riemannian Manifolds	143
7.5	Parallel Transport Along Horizontal Curves	145
7.6	The Curvature of a Connection	146
7.7	The Induced Curvature	148
7.8	The Metrical Connection	150
7.9	The Flat Connection	152

Cambridge University Press

978-0-521-89730-3 - Sub-Riemannian Geometry: General Theory and Examples

Ovidiu Calin and Der-Chen Chang

Table of Contents

[More information](#)

	Contents	ix
<b>8</b>	<b>Gauss' Theory of Sub-Riemannian Manifolds</b>	154
8.1	The Second Fundamental Form	154
8.2	The Adapted Connection	156
8.3	The Adapted Weingarten Map	160
8.4	The Variational Problem	163
8.5	The Case of the Sphere $\mathbb{S}^3$	168
	<b>Part II Examples and Applications</b>	
<b>9</b>	<b>Heisenberg Manifolds</b>	175
9.1	The Quantum Origins of the Heisenberg Group	175
9.2	Basic Definitions and Properties	176
9.3	Determinants of Skew-Symmetric Matrices	181
9.4	Heisenberg Manifolds as Contact Manifolds	182
9.5	The Curvature Two-Form	184
9.6	Volume Element on Heisenberg Manifolds	189
9.7	Singular Minimizers	195
9.8	The Acceleration Along a Horizontal Curve	197
9.9	The Heisenberg Group	199
9.10	A General Step 2 Case	202
9.11	Solving the Euler–Lagrange System with $\varphi(x)$ Linear	203
9.12	Periodic Solutions in the Case $\varphi(x)$ Linear	209
9.13	The Lagrange Multiplier Formula	211
9.14	Horizontal Diffeomorphisms	213
9.15	The Darboux Theorem	217
9.16	Connectivity on $\mathbb{R}^{2n+1}$	218
9.17	Local and Global Connectivity	224
9.18	$D$ -Harmonic Functions	226
9.19	Examples of $D$ -Harmonic Functions	229
<b>10</b>	<b>Examples of Heisenberg Manifolds</b>	231
10.1	The Sub-Riemannian Geometry of the Sphere $\mathbb{S}^3$	231
10.2	Connectivity on $\mathbb{S}^3$	237
10.3	Sub-Riemannian Geodesics: A Lagrangian Approach	241
10.4	Sub-Riemannian Geodesics: A Hamiltonian Approach	244
10.5	The Lie Group $SL(2, \mathbb{R})$	251
10.6	Liu and Sussman's Example	253
10.7	Skating and Car-Like Robots as Nonholonomic Models	256
10.8	An Exponential Example	263
<b>11</b>	<b>Grushin Manifolds</b>	271
11.1	Definition and Examples	271
11.2	The Geometry of Grushin Operator	273
11.3	Higher-Step Grushin Manifolds	279
11.4	A Step 3 Grushin Manifold	284

Cambridge University Press

978-0-521-89730-3 - Sub-Riemannian Geometry: General Theory and Examples

Ovidiu Calin and Der-Chen Chang

Table of Contents

[More information](#)

x	Contents	
	11.5 Another Grushin-Type Operator of Step 2	288
	11.6 Grushin Manifolds as a Limit of Riemannian Manifolds	296
<b>12</b>	<b>Hörmander Manifolds</b>	<b>302</b>
	12.1 Definition of Hörmander Manifolds	302
	12.2 The Martinet Distribution	303
	12.3 Engel's Group and Its Lie Algebra	314
	12.4 The Engel Distribution	316
	12.5 Regular Geodesics on Engel's Group	318
	12.6 Singular Geodesics on Engel's Group	327
	12.7 Geodesic Completeness on Engel's Group	329
	12.8 A Step 3 Rolling Manifold: The Rolling Penny	331
	12.9 A Step $2(k + 1)$ Case	344
	12.10 A Multiple-Step Example	346
<b>A</b>	<b>Local Nonsolvability</b>	<b>351</b>
<b>B</b>	<b>Fiber Bundles</b>	<b>354</b>
	B.1 Sub-Riemannian Fiber Bundles	354
	B.2 The Variational Problem	358
	B.3 The Hopf Fibration	360
	Bibliography	363
	Index	367