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978-0-521-89730-3 - Sub-Riemannian Geometry: General Theory and Examples
Ovidiu Calin and Der-Chen Chang
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Sub-Riemannian Geometry

General Theory and Examples

Sub-Riemannian manifolds are manifolds with the Heisenberg principle built in. This comprehensive text and reference begins by introducing the theory of sub-Riemannian manifolds using a variational approach in which all properties are obtained from minimum principles, a robust method that is novel in this context. The authors then present examples and applications, showing how Heisenberg manifolds (step 2 sub-Riemannian manifolds) might in the future play a role in quantum mechanics similar to the role played by the Riemannian manifolds in classical mechanics.

Sub-Riemannian Geometry: General Theory and Examples is the perfect resource for graduate students and researchers in pure and applied mathematics, theoretical physics, control theory, and thermodynamics interested in the most recent developments in sub-Riemannian geometry.

Ovidiu Calin is an associate professor of mathematics at Eastern Michigan University and a former visiting assistant professor at the University of Notre Dame. He received his Ph.D. in geometric analysis from the University of Toronto in 2000. He has written several monographs and numerous research papers in the field of geometric analysis and has delivered research lectures in several universities in North America, Asia, the Middle East, and Eastern Europe.

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Dedicated to
Professor Shing Tung Yau
on the occasion of his sixtieth birthday

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"I don't add hypotheses.
I derive everything from what is given."
Isaac Newton

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Preface

A few important discoveries in the field of thermodynamics in the 1800s made the first steps toward sub-Riemannian geometry. Carnot discovered the principle of an engine in 1824 involving two isotherms and two adiabatic processes, Jule studied adiabatic processes, and Clausius formulated the existence of the entropy in the second law of thermodynamics in 1854. In 1909 Carathéodory made the point regarding the relationship between the connectivity of two states by adiabatic processes and nonintegrability of a distribution, which is defined by the one-form of work. Chow proved the general global connectivity in 1934, and the same hypothesis was used by Hörmander in 1967 to prove the hypoellipticity of a sum of the squares of vector fields operators. However, the study of the invariants of a horizontal distribution, known as nonholonomic geometry, was initiated by the Romanian mathematician George Vranceanu in 1936.

The position of a ship on a sea is determined by three parameters: two coordinates x and y for the location and an angle to describe the orientation. Therefore, the position of a ship can be described by a point in a manifold. One can ask what is the shortest distance one should navigate to get from one position to another; this defines a Carnot–Carathéodory metric on the manifold $\mathbb{R}^2 \times \mathbb{S}^1$. In a similar way, a Carnot–Carathéodory metric can be defined on a general sub-Riemannian manifold. The study of sub-Riemannian geodesics is useful in determining the Carnot–Carathéodory distance between two points.

The study of the geometry of the Heisenberg group, which is the prototype of the sub-Riemannian geometry, was started by Gaveau in 1975. The understanding of the geometry of this group led Beals, Gaveau, and Greiner to characterize the fundamental solutions for heat-type subelliptic operators and Heisenberg sub-Laplacian operators in the 1990s. Meanwhile, many examples have been considered. Some of them have a behavior similar to the Heisenberg operator, but others do not. However, a unitary and general theory of these sub-Riemannian manifolds is still missing at the moment.

This book was written by the first author with the participation of the second. This work is mainly based on both the author's own recent research publications

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as well as a great deal of first author's unpublished work. It reflects the authors' best knowledge on the subject at the time it was written.

The main goal of Part I is to present a detailed analysis of the general theory of sub-Riemannian manifolds using Hamiltonian and Lagrangian formalism developed in the sub-Riemannian manifolds context. Other mathematical tools used are differential geometry, exterior differential systems, and the theory of elliptic functions.

Part II contains a rich collection of examples of sub-Riemannian manifolds of step 2 and higher, in which the computations can be done explicitly and a further precise study can be made. Each example involves different techniques, some of them involving elliptic integrals and hypergeometric functions. Some of these examples are computed here for the first time.

Why do we need a book on sub-Riemannian geometry? The authors believe the study of sub-Riemannian geometry helps with the understanding of subelliptic operators. A similar theory was developed between the Riemannian geometry and the elliptic operators. For instance, the heat kernel of a subelliptic operator depends on the geometry of the underlying horizontal distribution. It is known that for the case of bracket-generating distributions, any two points can be joined by piecewise horizontal curves. It is believed that the heat kernel is given by a path integral with respect to all horizontal curves joining the points x_0 and x in time t as in the formula $K(x_0, x; t) = \int_{\mathcal{PH}_{x_0,x;t}} e^{-S(\phi,t)}\, d\mathfrak{m}(\phi)$. Here $\mathcal{PH}_{x_0,x;t}$ denotes the space of horizontal curves between x_0 and x parameterized by $[0, t]$, $S(\phi, t)$ is the classical action along the horizontal curve $\phi \in \mathcal{PH}_{x_0,x;t}$, and $d\mathfrak{m}(\phi)$ is an analog of the Wiener measure along the horizontal distribution. The authors intend to return to these ideas in a future monograph.

An Overview for the Reader

The present work can be considered as a text for a course or seminars designed for graduate students interested in the most recent developments in sub-Riemannian geometry. It is useful for both pure and applied mathematicians and theoretical physicists working in the thermodynamics area. The goal of this book is to introduce the reader to the differential geometry of sub-Riemannian manifolds.

Scientific Outline

This book deals with the study of sub-Riemannian manifolds, which are manifolds with the Heisenberg principle built in. It is hoped that Heisenberg manifolds (step 2 sub-Riemannian manifolds) will play a role in quantum mechanics in the future, similar to the role played by the Riemannian manifolds in classical mechanics. Some people also speculate that superior-step sub-Riemannian manifolds may play a similar role in quantum field theory.

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Therefore it is important to understand sub-Riemannian as well as Riemannian manifolds. However, the sub-Riemannian manifolds behave very differently than Riemannian ones, and we need new methods and insights of investigation.

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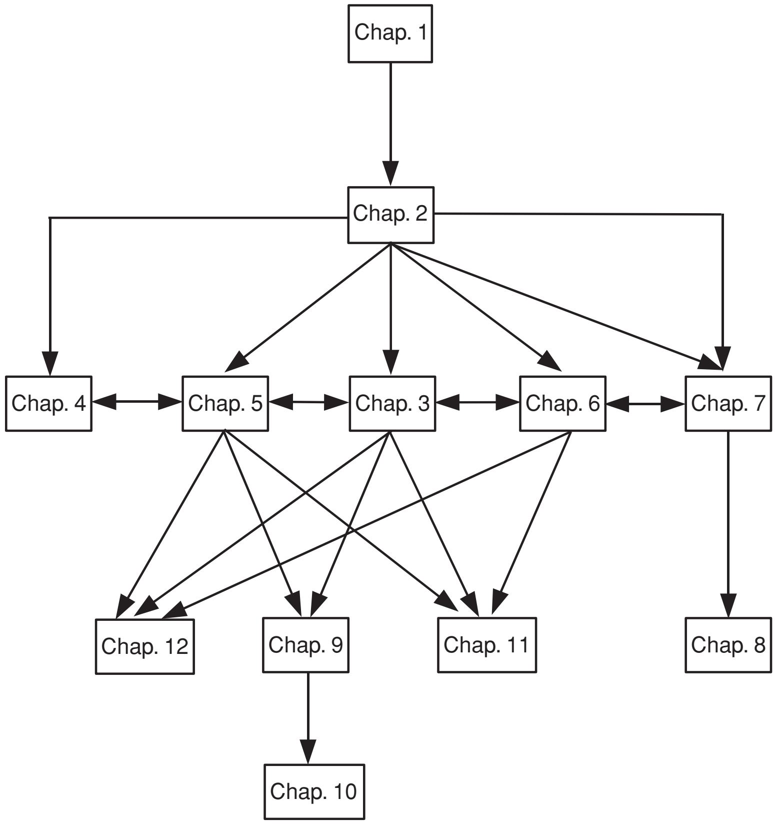

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