Elementary Differential Geometry

The link between the physical world and its visualisation is geometry. This easy-to-read, generously illustrated textbook presents an elementary introduction to differential geometry with emphasis on geometric results. Avoiding formalism as much as possible, the author harnesses basic mathematical skills in analysis and linear algebra to solve interesting geometric problems, which prepare students for more advanced study in mathematics and other scientific fields such as physics and computer science.

The wide range of topics includes curve theory, a detailed study of surfaces, curvature, variation of area and minimal surfaces, geodesics, spherical and hyperbolic geometry, the divergence theorem, triangulations, and the Gauss–Bonnet theorem. The section on cartography demonstrates the concrete importance of elementary differential geometry in applications. Clearly developed arguments and proofs, colour illustrations, and over 100 exercises and solutions make this book ideal for courses and self-study. The only prerequisites are one year of undergraduate calculus and linear algebra.

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Preface

This book evolved from courses about elementary differential geometry which I have taught in Freiburg, Hamburg and Potsdam. The word "elementary" should not be understood as "particularly easy", but indicates that the development of formalism, which would be necessary for a deeper study of differential geometry, is avoided as much as possible. We will instead approach geometrically interesting problems using tools from the standard fundamental courses in analysis and linear algebra. It is possible to raise interesting questions even about objects as "simple" as plane curves. The proof of the four-vertex theorem, for example, is anything but trivial.

The book is suitable for students from the second year of study onwards and can be used in lectures, seminars, or for private study.

The first chapter is interesting mostly for historical reasons. The reader can here find out how geometric results have been obtained from axioms for thousands of years, since Euclid. In particular, the controversy about the parallel axiom will be explained. In this chapter we will mostly follow Hilbert's presentation of plane geometry, since it is rather close to Euclid's formulation of the axioms and yet meets today's requirements for mathematical rigour. In the mean time the axiomatic system has been simplified significantly [2]. A presentation with only seven axioms can be found in [30].

Anyone who is only interested in differential geometry can begin with the second chapter. The theory of curves is developed here, with particular focus on curves in the plane and in three-dimensional space. Curvature, a central notion in differential geometry, appears for the first time. Of particular interest are global results, i.e. statements about the overall shape of closed curves. The above-mentioned four-vertex theorem and the theorems of Fenchel and Fáry–Milnor fall into this category. They tell us how much a space curve needs to curve so that it can close up (Fenchel) and how much a curve needs to curve to become knotted (Fáry–Milnor).

We begin to study surfaces in three-dimensional space in the third chapter. The necessary concepts are introduced, e.g. different notions of curvature, and some important classes of surfaces are studied in more detail. One of them is the class of minimal surfaces, which appear in nature as soap films. Some examples are illustrated in colour as well.

In the fourth chapter we change our point of view and concentrate on geometric quantities that can be obtained using measurements taken on the surface itself only. We study the shortest connecting curves between two points on a given surface, for example. This stance suggests the introduction of general Riemannian metrics, which allows us to construct new important geometries. The most prominent example is the hyperbolic plane, which, as Hilbert showed, cannot be realised as a "classical" surface. One reason why the hyperbolic plane is so important is that it ended the controversy about the parallel axiom. It is therefore often referred to as a non-Euclidean geometry. We devote ourselves to hyperbolic and spherical geometry and derive the most important trigonometric laws. Spherical geometry is used to discuss applications in cartography. We conclude the chapter with a comparison of different models of hyperbolic geometry illustrated by a woodcut of Dutch artist M. C. Escher.

In the fifth chapter we derive Gauss's divergence theorem and deduce that the total Gauss curvature of a closed surface does not depend on the Riemannian metric. The total curvature is thus a "topological invariant" of the surface.

The last chapter is dedicated to the topological interpretation of this quantity. We show that every compact surface can be triangulated, i.e. that it can be cut into triangles in a suitable way. The Gauss–Bonnet theorem then tells us that the total curvature can be found by counting vertices, edges and triangles. We conclude with the outlook and recommendations for further study.

Three appendices follow: first hints for solutions to the exercises, then a collection of useful formulae concerned with the inner geometry of surfaces and the most important trigonometric laws, and finally, the mathematical symbols used in this book are listed, to make it easier to look them up. As is the custom, the book ends with the references and the index.

The enumeration of theorems, lemmas, examples and so forth is done using three numbers, where the first one denotes the chapter and the second one the section. The numerous exercises are enumerated by two numbers, the first one being the chapter. They are interspersed in the text and mainly discuss examples, which can be used to practice the material treated so far. The further logical arguments do not build on most exercises, but reading and doing the exercises is recommended to establish the necessary familiarity with the introduced concepts.

At this point it is my pleasure to thank all those that made this book a success, e.g. by pointing out mistakes and making suggestions, including B. Ammann, M. Aubel, F. Auer, L. Außenhofer, P. Ghanaat, H. Karcher, A. Kreuzer, D. Lengeler, F. Pfäffle, C. Pries, W. Reichel, E. Schröder,

C. Schulz, T. Seidel, U. Semmelmann, H. Wendland, U. Witting, and U. Wöske. I am of course solely responsible for any mistakes in this book, which it will inevitably contain. I would be very grateful for a note telling me about them which can be sent to baer@math.uni-potsdam.de. Sincere thanks also go to Cambridge University Press, in particular C. Dennison, for her always pleasant and trusting collaboration.

This book has been typeset using LaTeX with the PSTricks package to produce the drawings. The coloured illustrations have been created with povray, the maps in the section on cartography with the Generic Mapping Tools for Unix. The transformations of Escher's woodcut to the Klein model of hyperbolic geometry on page 220 and to the half-plane mode on page 222 have been carried out with gimp using the mathmap plugin. I am very grateful to the developers of all this great open source software. The illustrations on pages 142, 206, and 256 have been created with Maple.

At last, a special thank you to A. Hornecker who contributed the coloured illustrations and many of the drawings and to P. Meerkamp who wrote a first translation of the German edition into English.

Notation

We here introduce some terms and conventions that will appear in this book again and again.

The cardinality of a set A is denoted using absolute value bars:

$$|A| = \text{number of elements of } A.$$

For the difference of two sets we write

$$A - B = \{x \in A \mid x \notin B\}.$$

$\mathbb{R}^n$ denotes the vector space of all column vectors with n real entries:

$$\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} = (x^1, \ldots, x^n)^\top \in \mathbb{R}^n.$$

The individual entries usually have their indices at the top. For a subset $A \subset \mathbb{R}^n$ the expression $\bar{A}$ denotes the closure, ∂A the boundary and $\mathring{A}$ the interior.

The Euclidean standard scalar product on $\mathbb{R}^n$ is written using angle brackets:

$$\left\langle (x^1, \ldots, x^n)^\top, (y^1, \ldots, y^n)^\top \right\rangle = \sum_{j=1}^{n} x^j y^j.$$

For a subspace $V \subset \mathbb{R}^n$

$$V^\perp = \{x \in \mathbb{R}^n \mid \langle x, y \rangle = 0 \text{ for all } y \in V\}$$

is the orthogonal complement.

The vector product on $\mathbb{R}^3$ is given by

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \times \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} = \begin{pmatrix} x^2y^3 - x^3y^2 \\ -x^1y^3 + x^3y^1 \\ x^1y^2 - x^2y^1 \end{pmatrix}.$$

For the real part of a complex number z we write $\Re(z)$ and for the natural logarithm of a positive real number x we write $\ln(x)$.

A *smooth* map denotes one that is infinitely often differentiable. For the differential or its Jacobian matrix at a point p we write

$$D_pF = \begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix},$$

where $F = (F^1, \ldots, F^m)^\top : \mathbb{R}^n \to \mathbb{R}^m$. For functions $f : \mathbb{R}^n \to \mathbb{R}$ specifically

$$\operatorname{grad} f = Df = \left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\right)^\top$$

denotes the gradient.

The group of invertible real $n \times n$ matrices is denoted by $\mathrm{GL}(n)$, the subgroup of orthogonal matrices by $\mathrm{O}(n)$:

$$\mathrm{O}(n) = \{A \in \mathrm{GL}(n) \mid A^\top A = \mathrm{Id}\},$$

and the subgroup of special orthogonal matrices by $\mathrm{SO}(n)$:

$$\mathrm{SO}(n) = \{A \in \mathrm{O}(n) \mid \det(A) = 1\}.$$

Here $A^\top$ denotes the transpose of A.