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1 Energy in planetary processes and the First Law of Thermodynamics

This book is about the physical chemistry of planetary processes. Although in detail each planetary body in the Solar System looks very different, all of the planets and moons have reached their current states as a result of the same fundamental laws of nature, which are codified into the sciences that we know as physics and chemistry. A real understanding of the nature and evolution of the bodies that make up the Solar System requires that we immerse ourselves in physics and chemistry, and that we come to think of planetary processes as specific applications of these sciences. These applications can be more complex, in the sense of the number of variables involved, than those that physicists and chemists deal with when working under controlled laboratory conditions. Perhaps for this reason students of geological and planetary sciences tend to view these sciences as separate or "stand alone". This is not so, however. Using an analogy that most of us are likely to be familiar with (and that, admittedly, may be a bit stretched), the sciences that we know as geology and planetary science (and their "sub-fields" such as petrology, mineralogy, or oceanography, to name just a few) are the "user interface", the set of graphics and icons and mnemonics that we see on our computer screens. This user interface is supported and made possible by a rich and complex operating system (e.g. Linux, Windows, Mac, according to our tastes). The "operating system" of geology and planetary sciences consists of physics and chemistry. By immersing ourselves in the "operating system" we will be able to see connections among planetary processes that we might not have suspected, and we will be able to better understand what makes each planetary body unique.

We have talked about planetary processes – but exactly what is a planetary process? Simply stated, anything in a planet, physical or chemical, that changes with time ("geological", and also "biological", are specific instances of "physical or chemical"). Planetary processes must be driven by some source of energy, otherwise they would stop and cease to be processes. We can furthermore make the general statement that, the higher the rate of energy supply, the more active a planetary process, and the system in which it occurs, are. Our first task, then, is to formalize our understanding of energy and to lay out the mathematical framework that makes it possible to track energy conversion processes.

In this chapter we will examine the ways in which energy is manifested in planetary processes. Along the way, we shall introduce a number of fundamental physical concepts and tools that must become part of our physical intuition and operating practice as we attack problems in planetary sciences. The chapter culminates with a formal development of the First Law of Thermodynamics and a discussion of some of the relationships between macroscopic phenomena and their microscopic underpinnings.

1.1 Some necessary definitions

It is customary to start books on thermodynamics by devoting several sections to defining much of the terminology that will be used throughout the work. It is my experience that if one defines all terms at the beginning but some of those terms are not used until much later, the meaning of many of these "deferred" terms does not become clear until they are applied in a specific example. Some backtracking necessarily ensues, with some loss of efficiency (energy dissipation, if we were defining that term here!) and, worse, with a tendency to forget the rigorous meaning of the terms. I have thus chosen to define terms as their need is first encountered, so that definitions will always be given in specific contexts that will make their meaning clear and easy to assimilate. There is, however, a minimum set of definitions that we must consider at this point.

Thermodynamics is the subset of physics that studies energy transformation processes, and in particular processes in which thermal energy is involved. As we shall see, some of the manifestations of thermal energy may not entail changes in temperature or exchanges of heat, so the meaning of thermodynamics is more subtle than what this definition may suggest.

In order to make problems mathematically tractable, it is customary when studying thermodynamics to subdivide the universe into systems. We will then call that part of the universe that we are studying the *system*, and the surrounding parts of the universe with which our system may be able to interact becomes the *environment.* We are usually free to define our system in any way we please, so that we can tailor it to the problem that we are trying to understand. For example, our system may be a mineral assemblage within a thin section, a parcel of volcanic gas expanding during an eruption, a magma chamber, a planetary atmosphere or a planetary core. Systems may be *open* or *closed*, depending on whether or not matter can move across their boundaries. Of course, most systems in nature are open to some extent. The basic thermodynamic relationships, however, are most easily introduced and developed for closed systems. Throughout this book, unless we explicitly state that we are considering an open system, it must be understood that we are dealing with closed systems. The terms "open" and "closed" refer **only** to mass transfer and **not** to energy transfer. Energy transfer can take place across the boundaries of both closed and open systems.

Classical thermodynamics concerns itself with systems that are at *equilibrium*. A rigorous definition of thermodynamic equilibrium is necessary, but cannot be implemented until we have developed much of the mathematical formalism of thermodynamics. For now, we will state that, in order for a closed system in which gravity can be ignored to be at equilibrium, its temperature and pressure must be uniform and if, in addition, there is no energy transfer across its boundaries, then the amounts and compositions of physically distinguishable subsets of the system (that we will call *phases*) must not change with time. The effect of gravity cannot be ignored over planetary length-scales, however, and is discussed in Chapters 2, 3 and 13. This definition is *not* complete but it is consistent with the formal definition of chemical equilibrium and will allow us to "feel our way" around thermodynamics until we can construct that formal definition (in Chapter 4). An example may clarify our intuitive definition. Suppose that our system consists of a certain amount of liquid water and a certain amount of ice, each of which is a different phase, held in a container that is a perfect thermal insulator. This is an example of a *heterogeneous* system, because it has more than one distinguishable part or phase. If the pressure and temperature everywhere in the system are 1 bar and 0 °C, then the system is at equilibrium because the relative amounts of ice and water will not change with time. If the temperature and the pressure are any other combination of values, then the system is *not* at equilibrium, because as time goes by the amount of one of the phases will increase at the expense of the other. If the temperatures of the phases are different (e.g. we open the container and dump ice at −20 °C into water at 20 °C, then close the container), then the system is also not at equilibrium because one of the phases will grow at the expense of the other. Let us assume that the relative amounts of ice and water are such that all of the water in the container freezes. We now have a *homogeneous system*, i.e., one that consists of a single phase. This system will be at equilibrium only once its temperature is uniform and heat flow within the system ceases. In general, a homogeneous system is at equilibrium if there are no gradients in temperature, pressure nor composition (Chapters 4 and 12), although in the presence of an external field, such as a gravitational field, this requirement must be relaxed (Chapter 13).

We will often make reference to the *state* of a system. The implication when we do so, unless we say otherwise, is that we refer to the state of a system at equilibrium. The state of a system at equilibrium can be fully characterized by the values of a small number of variables, of which pressure and temperature are the most familiar ones. If we have a homogeneous system, for example a given amount of liquid water, then the state of the system is fully characterized once we specify its pressure, P, and temperature, T. For every combination of pressure and temperature liquid water has a single and well defined set of values for its physical properties, such as density, ρ (or its inverse, molar volume, V), refractive index or dielectric constant. What this says is that we only need to specify P and T to specify the state of the system. In principle, we can specify the state of the system equally well by choosing another pair of variables, such as molar volume and refractive index. Thermodynamics allows us to do this (the reasons will become clear in Chapter 6), even if it may not be the most sensible choice. For more complex systems we may need additional variables, but whether this is the case, and how many more variables we need, is not intuitively obvious (again, we will develop this formally in Chapter 6). For now, we note that if we go back to the system consisting of ice and water in an insulated container, we need just two variables (P and T) to specify the state of that system. The proportion of the two phases does not affect the thermodynamic state of the system as long as it is at equilibrium, even though it may be important in other contexts.

One final subject that must be covered in this introductory section is that of the thermodynamic temperature scale. Temperature is a fundamental physical quantity, in the sense that it is irreducible to a combination of simpler quantities. Other fundamental quantities are length, mass, time, and electric charge. The units in which these quantities are measured are defined in terms of conventional values such as the meter, kilogram and second. The absolute nature of these units is immediately evident because it is easy to grasp, at least in principle, what we mean by zero length, zero mass or zero time, and because it is also self evident that these three fundamental dimensions cannot take negative values. Temperature is different, because the temperature scales that are used in everyday life, and in many engineering applications, are based on arbitrary references which do not establish absolute values and, in particular, give no special meaning to the value of zero. In the Celsius or centigrade scale, zero is the temperature at which ice and water are at equilibrium at 1 bar pressure, and in this scale negative temperatures are obviously possible. An absolute temperature scale exists but is not easy to define until we have studied thermodynamics in some depth. We shall not worry here about how absolute thermodynamic temperatures are defined, but good discussions can be found, for example, in the classic textbooks by Lewis and Randall (1961), Glasstone (1946) and Guggenheim (1967). What we will do is

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emphasize that *all thermodynamic calculations must be carried out in this absolute temperature scale*, in which temperatures are measured in kelvins (symbolized K). Conversion is accomplished by adding 273.15 to the temperature in ∘C in order to obtain the temperature in K (note no "°" in K!).

1.2 Conservation of energy and different manifestations of energy

Conservation of energy (or, more accurately, mass-energy) is an example of a law of nature. By a "law of nature" we mean a statement that summarizes a large number of empirical observations (large enough that we are confident that we will not come across a contradictory observation) and that cannot be derived from simpler concepts, principles or laws. It is just a statement of a specific way in which our universe works. Whether or not it may be possible to understand why our universe works as it does is the subject of considerable argument among physicist working at the frontiers of knowledge, but it is a topic that is beyond the scope of this book. One possible statement of the law of conservation of energy is that any change in the total energy content of a system must equal the amount of energy received by the system minus the amount of energy extracted from the system. In order to be useful, however, a law of nature must be expressed as a mathematical statement. Why? Because mathematics is the only unambiguous and universally intelligible language. The language of mathematics has the additional advantages of being economical (i.e. concepts are expressed with the minimum possible number of symbols) and of being accompanied by a well-defined set of operation and manipulation rules. The First Law of Thermodynamics, which we will introduce in Section 1.10, is the mathematical statement of the law of conservation of energy and the starting point for our thermodynamic exploration of planetary processes.

Implicit in the law of conservation of energy is the concept that there are different kinds of energy that are equivalent to each other. Equivalence, however, does not mean unrestricted convertibility, leading to the concept that there are two fundamentally different manifestations of energy. There are those kinds of energy that can be fully and freely converted to other kinds. This category comprises all types of energy but one. Examples include: mechanical energy (in its various manifestations), electric energy, electromagnetic energy, and relativistic rest mass. Thermal energy is the one type of energy that belongs in a different category because it cannot be freely nor fully converted to other types of energy *although unrestricted conversion of other types of energy to thermal energy is always possible*. This distinction between thermal and other types of energy is at the root of another law of nature, called the Second Law of Thermodynamics. In Chapter 4 we will examine this law, which is perhaps one of the most fundamental, and mysterious, laws of nature. In the meantime, we need to formalize the definitions and mathematical formulations of the various types of energy that are important in planetary processes.

1.3 Mechanical energy. An introduction to dissipative and non-dissipative transformations

Although we all have an intuitive knowledge of what energy is, it is a concept that is notoriously difficult to define. For example, in elementary classical physics we learn that "energy is the capability to do work". You will have to make up your own mind on whether or not this definition is useful (or, indeed, whether it is a definition at all!). The concept of work, in contrast, has a unique and simple mathematical definition which, in differential form, is:

$$d\boldsymbol{W} = \bar{\boldsymbol{f}} \cdot d\bar{\boldsymbol{x}}, \tag{1.1}$$

where W is work, $\bar{f}$ is force and $d\bar{x}$ is the distance over which the point of application of the force moves. The use of lowercase bold symbols with an overstrike for $\bar{f}$ and $d\bar{x}$ means that these two quantities are vectors, and the dot between the two vectors represents a mathematical operation called "inner product" or "dot product" (Box 1.1). The inner product of two vectors yields a scalar quantity (W in this case), according to equation (1.1.1).

Box 1.1 **Vectors, fields and the inner product**

Physical magnitudes that have orientation, such as force and distance, are represented by geometric objects that are loosely called vectors. There are, in fact, two distinct types of vectors. Distance is an example of what is called a contravariant vector, and force is an example of a covariant vector. In modern mathematical language, distance is a *vector* and force is a *one-form*. Although both are oriented geometrical objects, one difference between the two is that one-forms are gradients of scalar fields, and vectors are not. A field is a mathematical function that gives the value of a variable as a function of space and time. If the variable is a scalar, such as temperature or density, the field is called a scalar field and the function returns a single number at each point in space and time. Suppose now that we keep time fixed. We can then determine the rate of change of the field intensity (e.g. temperature) relative to each of the three orthogonal spatial directions (see also Box 1.3). The set of the three derivatives ($\partial T/\partial x$, $\partial T/\partial y$, $\partial T/\partial z$) is the gradient of the temperature field. This geometrical object is a one-form. Just as a vector, it has orientation (which is the direction of maximum rate of change) but it differs from a vector, among other things, in that its magnitude is given not by a length but by the separation between contour lines. The more closely spaced the contour lines are, the greater the magnitude of the one-form is. Force is the gradient of a potential energy field. In particular, the gravitational force is the gradient of the gravitational potential energy field. The more steeply gravitational potential energy varies, the stronger the gravitational force is. Excellent and comprehensive introductions to these concepts are given in the first chapters of the massive text *Gravitation* by Misner, Thorne and Wheeler (1973), and in Burke (1985). A classical and very accessible exposition (using the terminology of contravariant and covariant vectors, rather than vectors and one-forms) is given by Kreyszig (1991).

The magnitude of a vector or a one-form is a scalar that measures its "intensity" and is symbolized by $|x|$, where $\bar{x}$ is the vector or one-form. The magnitude of a vector corresponds to the intuitive concept of length, but, as I mentioned above, the magnitude of a one-form is better thought of as the spacing between contour lines – the more closely spaced the contour lines are, the greater the magnitude of the one-form (i.e. the steeper the gradient). The inner product is an operation that combines a vector and a one-form and returns a scalar. Geometrically, the inner product of $\bar{x}$ and $\bar{y}$ is the scalar A defined by:

$$A = \bar{x} \cdot \bar{y} \equiv |x||y| \cos\theta, \tag{1.1.1}$$

where θ is the angle between the two objects. Thus, if $\bar{x}$ is a vector oriented perpendicular ($\theta = \pi/2$) to the gradient of a scalar field ($=$ the one-form $\bar{y}$), the inner product vanishes. The inner product attains its maximum absolute value if the vector points in the direction of the field gradient ($\theta = 0$) or exactly opposite ($\theta = \pi$). It is positive if $0 \leq \theta < \pi/2$ and negative for $\pi/2 < \theta \leq \pi$.

We can use the concept of work to turn the definition of energy on its head and in the process make it clearer. Whenever work is performed, there is a force interaction between a system and its environment, or between different parts of a system. Work is responsible for energy transfer between the objects that interact, so some of these objects give up energy, and the same amount of energy, *measured by the magnitude of the work performed*, is stored in others. This statement may not be a definition of energy, but at least it tells us how to calculate changes in energy content. From this statement we also rescue the fact that energy has the same dimension as work (Box 1.2).

Box 1.2 Dimensional analysis

The dimension of a physical quantity is a fundamental and immutable property that defines what the quantity is. Thus, distance has dimension of length, inertia has dimension of mass, and duration has dimension of time. Units are arbitrary scales that are used to measure the magnitude of a physical quantity, for instance, distance can be measured in meters, kilometers, parsecs, etc. These are different units that measure the dimension length. Some physical quantities are fundamental in the sense that their dimension cannot be reduced to combinations of other dimensions. Examples are length, mass, electric charge, time and temperature. The key idea of dimensional analysis is that in any equation relating physical quantities the identity applies to dimension as well. The fundamental dimensions length, mass, electric charge, time and temperature are labeled $[L]$, $[M]$, $[Q]$, $[T]$ and $[\Theta]$, respectively. Dimensions of derived physical quantities are reduced to combinations of these fundamental dimensions. For instance, acceleration has dimension $[L]\times[T]^{-2}$, and force has dimension $[M]\times[L]\times[T]^{-2}$.

In the notation of dimensional analysis, enclosing the name or symbol of a quantity in square brackets means that we are referring to the dimension of the quantity. From equation (1.1) we have:

$$[work] = \left[force\right] \times [distance] \tag{1.2.1}$$

or:

$$[work] = [M] \times [L]^2 \times [T]^{-2}. \tag{1.2.2}$$

The units of the fundamental dimensions in the SI system are meter (m, length), kilogram (kg, mass), coulomb (C, electric charge), second (s, time) and kelvin (K, temperature). The SI unit of force is the newton (N), and from dimensional analysis we see that $1\ \mathrm{N} = 1\ \mathrm{kg\ m\ s^{-2}}$. Similarly, the SI unit of work, or energy, is the joule (J) and $1\ \mathrm{J} = 1\ \mathrm{N\ m} = 1\ \mathrm{kg\ m^2\ s^{-2}}$. Note the subtle but important difference between the concepts of dimension and units. The dimension of a quantity is unique and universal, for instance, $[M]\times[L]^2\times[T]^{-2}$ for work or energy. The units can be anything we agree upon, as long as they conform to the required dimensional equation, such as (1.2.2). In general I will use SI units throughout this book, with one important exception, which is that I use bars and its multiples and submultiples (kbar, Mbar, mbar, etc.) as the unit of pressure, instead of pascal (Pa), which is the SI unit. The conversion factor is $1\ \mathrm{bar} = 10^5\ \mathrm{Pa}$.

Just as it is difficult to define energy, it is also somewhat unsatisfactory to try to pigeonhole different types of energy into strict categories. In this and subsequent chapters we will come across examples that will highlight this difficulty. As a matter of convenience and tradition, however, we will include in our discussion of mechanical energy only the energy associated with motion (kinetic energy) and that one associated with position in a gravitational field (gravitational potential energy). Both of these types of energy play crucial roles in planetary processes. We must keep in mind, however, that many other types of energy, such as the energy associated with a change in shape or size of an object, the energy associated with a magnetic or electrostatic field, or the energy contained in a crystalline lattice, can ultimately be reduced to specific manifestations of mechanical energy. The one type of energy that is distinct is heat, and in this section we will begin our journey towards our understanding of the reasons for, and consequences of, this difference – this is, indeed, what much of thermodynamics is all about.

1.3.1 Gravitational potential energy

Gravitational potential energy is perhaps the most familiar kind of mechanical energy – it is responsible, for instance, for the fact that it is harder to hike up a mountain than down. Gravitational potential energy arises from the existence of a gravitational attractive force that acts between objects with mass. The magnitude of the gravitational force $|\bar{f}_g|$ between two bodies with masses m_1 and m_2 separated by a distance x is given by Newton's law of universal gravitation:

$$|\bar{\boldsymbol{f}}_g| = \frac{Gm_1m_2}{x^2}, \tag{1.2}$$

where G is the universal gravitational constant (physical constants and other important numerical data are given in Appendix 1). Force is a vector (more precisely, a covariant vector or one-form, Box 1.1) and this expression yields only its magnitude. The gravitational attraction caused by a body is described by a *vector field*, which is a function that assigns a vector to each point of space. The magnitude of the vector, called the field intensity, is the gravitational force per unit mass, i.e. the gravitational acceleration. The orientation of the vector depends on the distribution of mass in the body that generates the field. For a point-like mass it is oriented towards the mass.

Let us consider an object of mass m, say a rock, in the gravitational field of another object, for instance a planet, of mass M. If the rock experiences a displacement $d\bar{x}$, the gravitational force $\bar{f}_g$ performs an amount of work given by equation (1.1). Gravitational potential energy, U_g, depends only on the position of an object in a gravitational field. The work performed by the gravitational force represents energy that is extracted from the object's gravitational potential energy. Because of the law of conservation of energy the changes in the two variables must balance out, which in differential form we write as follows:

$$d\boldsymbol{U}_g + \bar{\boldsymbol{f}}_g \cdot d\bar{\boldsymbol{x}} = 0. \tag{1.3}$$

We consider a displacement that is directed radially outwards from the planet with mass M (Fig. 1.1). If we define the direction away from the planet as being the positive orientation,

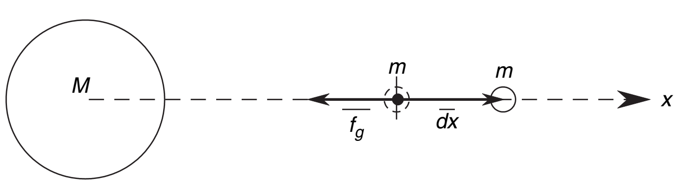

Fig. 1.1 Work performed by the gravitational force $\bar{f}_g$ between two bodies with masses M and m, when body m moves a distance $d\bar{x}$ towards infinity.

then $\bar{f}_g$ always has a negative orientation – this is the mathematical expression of the fact that gravity is always an attractive force. We can then write the inner product in equation (1.3) as:

$$\bar{\boldsymbol{f}}_g \cdot d\bar{\boldsymbol{x}} = -\frac{GMm}{x^2}dx, \tag{1.4}$$

where the negative sign arises from the fact that we are calculating the inner product between a vector and a one-form that subtend an angle of 180 ° (Box 1.1). This is the work *performed by the gravitational force* (the general three-dimensional case requires more complex notation, but the physics are well summarized in (1.4)). The change in gravitational potential energy that corresponds to a finite displacement between two positions, x_a and x_b can be obtained by substituting (1.4) into (1.3) and integrating:

$$\boldsymbol{U}_{g,b} - \boldsymbol{U}_{g,a} = -\int_a^b \bar{\boldsymbol{f}}_g \cdot d\bar{\boldsymbol{x}} = GMm\int_a^b \frac{dx}{x^2} = -GMm\left(\frac{1}{x_b} - \frac{1}{x_a}\right). \tag{1.5}$$

By convention, we define gravitational potential energy as being 0 when the objects are separated by an infinite distance, i.e. we make $U_{g,a} = 0$ as $x_a \rightarrow \infty$. With this convention we then define the gravitational potential energy of an object with mass m in the gravitational field of a planet with mass M as:

$$\boldsymbol{U}_g = -\frac{GMm}{r}, \tag{1.6}$$

where r is the distance between the centers of mass of the two bodies. For any finite value of r, U_g is negative, and U_g approaches a maximum value of 0 as the separation between the two bodies approaches infinity. When writing equation (1.4) I justified the negative sign on purely mathematical grounds, as arising from the inner product of two oppositely pointing vectors, but we can now see the physical meaning of this negative sign. Suppose mass m is moved away from M ($\Delta r > 0$). In order for this to occur an external agent must perform work. By conservation of energy this work becomes stored as potential energy in the gravitational field. Therefore it must be $\Delta U_g > 0$, which is what results from equation (1.6) with $\Delta r > 0$. Conversely, if $\Delta r < 0$ the gravitational field gives up energy ($\Delta U_g < 0$) which is transferred to mass m and appears, for example, as kinetic energy.

I will introduce here two other equations that we use in the analysis of gravitationally driven planetary heating in Chapter 2. First, we can see from equation (1.2) that the gravitational acceleration, g, due to a body of mass M at a distance r from its center of mass is given by:

$$g = -\frac{GM}{r^2}, \tag{1.7}$$

where the negative sign expresses the fact that gravitational acceleration is always attractive (directed towards the mass that causes it, where r is positive away from the mass). Recall that the numerical value of g is the intensity of the gravitational field. We also define the gravitational potential, Φ_g, as the gravitational potential energy per unit mass:

$$\Phi_g = -\frac{GM}{r}. \tag{1.8}$$

Gravitational potential is a scalar field (Box 1.1). Gravitational acceleration is a one-form (or covariant vector) that is the gradient of the potential field. The magnitude of the one-form is given by:

$$g = -\frac{d\Phi_g}{dr}. \tag{1.9}$$

Force is another one-form, that is the product of a scalar (mass) times acceleration (Newton's second law of motion). Mass, or inertia, is the scaling factor between force and acceleration; this is the origin of the term *scalar*.

We are generally concerned with *differences* in gravitational potential energy between different states of a system. For example, when tectonic processes elevate a mountain range, or when lava flows build a volcano, gravitational potential energy is stored in the rocks. How much potential energy is stored in a mountain range? This depends on the distance that the rocks can move towards the center of the planet before they get to a level below which they can move no further. How do we define such a level? Sea level may be a good first approximation, but we can give a more general answer, that will also hold for planets without oceans. We begin by looking at two additional questions. First, where did this gravitational potential energy associated with topography come from? Conservation of energy requires that we identify an energy source, which in this case entails conversion of some of the planet's internal heat to gravitational potential energy (more on this in Chapter 3). Second, what happens to this potential energy as the mountain loses elevation? The short answer is that this gravitational potential energy ultimately becomes heat and is dissipated to space, but the pathway may entail some intermediate steps, depending on how the mountain loses elevation. In general, elevation is lost by a combination of three processes: erosion, isostatic adjustment and tectonic collapse. During erosion, potential energy becomes heat as a result of friction during sediment transport and also when particles come to their final resting place in a sedimentary basin. Isostatic adjustment may return gravitational potential energy to the mantle, either as heat or as mechanical energy. Tectonic collapse results in gravitational potential energy either being dissipated directly as heat during ductile deformation, or being stored as elastic energy (another type of mechanical energy that we will discuss) to be eventually dissipated, ultimately as heat too, during earthquakes. All of these processes drive towards converting the surface of the planet to one over which there are no differences in gravitational potential energy. Such a surface is called an *equipotential surface*. A well-defined reference level for potential energy on Earth is thus the *geoid*, which is defined as the equipotential surface that is as close as possible (e.g. in a least square sense) to mean sea level (see Worked Example 1.1). In planets without oceans, we may choose as our reference the equipotential surface that is as close as possible to the mean planetary radius (and, if we were to follow the same convention used for Earth, we should call such surfaces: areoid, aphrodoid, hermoid, selenoid, etc.). We will generally be concerned with differences between the value of U_g at the geoid and its value at any other level that we may be interested in.

Worked Example 1.1 Gravitational potential energy and topography

(a) Consider a mass m of rock initially located on Earth's geoid. Let R be the geoid's mean radius. The rock is then moved to an elevation h above the geoid, such that $h \ll R$, i.e. we stay close to the planet's surface. The initial distance between the two centers of mass (the

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rock's and the Earth's) is R, and the final distance, after the rock is raised, is $R + h$. We first use equation (1.6) to show that the gravitational potential energy of the rock in its final state, relative to the geoid, is given by mgh, the equation that you probably remember from introductory physics.

Calling the gravitational potential energies at the geoid and at an elevation h above the geoid $\boldsymbol{U}_{g,geoid}$ and $\boldsymbol{U}_{g,h}$, we have:

$$\boldsymbol{U}_{g,h} - \boldsymbol{U}_{g,geoid} = -\frac{GMm}{R+h} - \left(-\frac{GMm}{R}\right) = \frac{GMmh}{R(R+h)}. \tag{1.10}$$

If we stay close to the planet's surface, then $R(R+h) \approx R^2$. We can also consider the planet's gravitational acceleration to be constant over the interval R to $R + h$. Using equation (1.7) to calculate gravitational acceleration at the geoid and substituting in (1.10):

$$\boldsymbol{U}_{g,h} - \boldsymbol{U}_{g,geoid} \approx \frac{GMmh}{R^2} = -mgh = m|g|h. \tag{1.11}$$

With our sign convention g is always a negative quantity. The "g" in mgh is thus the magnitude of g, as shown in the last term of equation (1.11).

(b) The Sierra Nevada of California is the largest uninterrupted mountain range in the United States. It is roughly 600 km long, 100 km wide and has a mean elevation of 1.5 km (averaged over this horizontal extent). Assuming that this average elevation represents the center of mass of the mountain range, and that the rocks making up the Sierra Nevada have an average density of 2800 kg m^{-3}, what is the potential energy stored in the Sierra Nevada relative to the geoid? At the geoid, $g \approx 9.8$ m s^{-2}. Plugging in these values into equation (1.11) we find that the Sierra Nevada stores approximately 3.7×10^{21} J of gravitational potential energy, approximately equivalent to an explosive yield of one million megatons (see Section 1.12.2).

(c) Uplift of the Sierra Nevada has occurred over the past 5 million years. Assuming a constant rate of uplift over that time interval, the energy flux (energy per unit area per unit time) that went into elevating the Sierra Nevada is approximately 3.9×10^{-4} W m^{-2} = 0.39 mW m^{-2}, where 1 W(Watt) = 1 J s^{-1}. Typical terrestrial heat fluxes are of the order of 50–100 mW m^{-2}, i.e. two orders of magnitude greater. There is plenty of energy in the Earth to elevate mountain ranges.

1.3.2 Kinetic energy

Bodies in motion have kinetic energy, $\boldsymbol{U}_k$, that arises from their speed and is given by:

$$\boldsymbol{U}_k = \frac{1}{2}mv^2, \tag{1.12}$$

where v is the magnitude of the body's velocity, i.e. its speed. Kinetic energy is a function of the *relative* speed between a body and an observer. For example, if we observe, from a location at rest on the Earth, an asteroid of mass m moving towards Earth with speed v, the kinetic energy of the asteroid in our reference frame is given by equation (1.12) and the Earth has no kinetic energy in our reference frame. Measured from a reference