How to Think Like a Mathematician

Looking for a head start in your undergraduate degree in mathematics? Maybe you've already started your degree and feel bewildered by the subject you previously loved? Don't panic! This friendly companion will ease your transition to real mathematical thinking.

Working through the book you will develop an arsenal of techniques to help you unlock the meaning of definitions, theorems and proofs, solve problems, and write mathematics effectively. All the major methods of proof – direct method, cases, induction, contradiction and contrapositive – are featured. Concrete examples are used throughout, and you'll get plenty of practice on topics common to many courses such as divisors, Euclidean Algorithm, modular arithmetic, equivalence relations, and injectivity and surjectivity of functions.

The material has been tested by real students over many years so all the essentials are covered. With over 300 exercises to help you test your progress, you'll soon learn how to think like a mathematician.

Essential for any starting undergraduate in mathematics, this book can also help if you're studying engineering or physics and need access to undergraduate mathematics topics, or if you're taking a subject that requires logic such as computer science, philosophy or linguistics.

How to Think Like a Mathematician

A Companion to Undergraduate Mathematics

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To Mum and Dad – Thanks for everything.

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Question: How many months have 28 days?
Mathematician's answer: All of them.

The power of mathematics

Mathematics is the most powerful tool we have. It controls our world. We can use it to put men on the moon. We use it to calculate how much insulin a diabetic should take. It is hard to get right.

And yet. And yet . . . And yet people who use or like mathematics are considered geeks or nerds.[1] And yet mathematics is considered useless by most people – throughout history children at school have whined 'When am I ever going to use this?'

Why would anyone want to become a mathematician? As mentioned earlier mathematics is a very powerful tool. Jobs that use mathematics are often well-paid and people do tend to be impressed. There are a number of responses from non-mathematicians when meeting a mathematician, the most common being 'I hated maths at school. I wasn't any good at it', but another common response is 'You must be really clever.'

The concept

The aim of this book is to divulge the secrets of how a mathematician actually thinks. As I went through my mathematical career, there were many instances when I thought, 'I wish someone had told me that earlier.' This is a collection of such advice. Well, I hope it is more than such a collection. I wish to present an attitude – a way of thinking and doing mathematics that works – not just a collection of techniques (which I will present as well!)

If you are a beginner, then studying high-level mathematics probably involves using study skills new to you. I will not be discussing generic study skills necessary for success – time management, note taking, exam technique and so on; for this information you must look elsewhere.

I want you to be able to think like a mathematician and so my aim is to give you a book jam-packed with practical advice and helpful hints on how to acquire skills specific to

[1] Add your own favourite term of abuse for the intelligent but unstylish.

thinking like a mathematician. Some points are subtle, others appear obvious when you have been told them. For example, when trying to show that an equation holds you should take the most complicated side and reduce it until you get to the other side (page 143). Some advice involves high-level mathematical thinking and will be too sophisticated for a beginner – so don't worry if you don't understand it all immediately.

How to use this book

Each part has a different style as it deals with a different idea or set of ideas. The book contains a lot of information and, like most mathematics books, you can't read it like a novel in one sitting.

Some friendly advice

And now for some friendly advice that you have probably heard before – but is worth repeating.

- *It's up to you* – Your actions are likely to be the greatest determiner of the outcome of your studies. Consider the ancient proverb: The teacher can open the door, but you must enter by yourself.
- *Be active* – Read the book. Do the exercises set.
- *Think for yourself* – Always good advice.
- *Question everything* – Be sceptical of all results presented to you. Don't accept them until you are sure you believe them.
- *Observe* – The power of Sherlock Holmes came not from his deductions but his observations.
- *Prepare to be wrong* – You will often be told you are wrong when doing mathematics. Don't despair; mathematics is hard, but the rewards are great. Use it to spur yourself on.
- *Don't memorize – seek to understand* – It is easy to remember what you truly understand.
- *Develop your intuition* – But don't trust it completely.
- *Collaborate* – Work with others, if you can, to understand the mathematics. This isn't a competition. Don't merely copy from them though!
- *Reflect* – Look back and see what you have learned. Ask yourself how you could have done better.

To instructors and lecturers – a moment of your valuable time

One of my colleagues recently complained to me that when a student is given a statement of the form A implies B to prove their method of proof is generally wholly inadequate. He jokingly said, the student assumes A, works with that for a bit, uses the fact that B is true and so concludes that A is true. How can it be that so many students have such a hard time constructing logical arguments that form the backbone of proofs?

I wish I had an answer to this. This book is an attempt at an answer. It is not a theoretical manifesto. The ideas have been tried and tested from years of teaching to improve mathematical thinking in my students. I hope I have provided some good techniques to get them onto the path of understanding.

If you want to use this book, then I suggest you take your favourite bits or pick some techniques that you know your own students find hard, as even I think that students cannot swallow every piece of advice in this book in a single course. One aim in my own teaching is to be inspirational to students. Mathematics should be exciting. If the students feel this excitement, they are motivated to study and, as in the proverb quoted above, will enter by themselves. I aim to make them free to explore, give them the tools to climb the mountains, and give them their own compasses so they can explore other mathematical lands. Achieving this is hard, as you know, and it is often not lack of time, resources, help from the university or colleagues that is the problem. Often, through no fault of their own, it is the students themselves. Unfortunately, they are not taught to have a questioning nature, they are taught to have an answering nature. They expect us to ask questions and for them to give the answers because that is they way they have been educated. This book aims to give them the questions they need to ask so they don't need me anymore.

I'd just like to thank . . .

This book has had a rather lengthy genesis and so there are many people to thank for influencing me or my choice of contents. Some of the material appeared in a booklet of the same name, given to all first-year Mathematics students at the University of Leeds, and so many students and staff have given their opinions on it over the years. The booklet was available on the web, and people from around the world have sent unsolicited comments. My thanks go to Ahmed Ali, John Bibby, Garth Dales, Tobias Gläßer, Chris Robson, Sergey Klokov, Katy Mills, Mike Robinson and Rachael Smith, and to students at the University of Leeds and at the University of Warwick who were first subjected to my wild theories and experiments (and whose names I have forgotten). Many thanks to David Franco, Margit Messmer, Alan Slomson and Maria Veretennikova for reading a preliminary draft. Particular thanks to Margit and Alan with whom I have had many fruitful discussions. My thanks to an anonymous referee and all the people at the Cambridge University Press who were involved in publishing this book, in particular, Peter Thompson.

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