HAMILTONIAN MECHANICS OF GAUGE SYSTEMS

The principles of gauge symmetry and quantization are fundamental to modern understanding of the laws of electromagnetism, weak and strong subatomic forces, and the theory of general relativity. Ideal for graduate students and researchers in theoretical and mathematical physics, this unique book provides a systematic introduction to Hamiltonian mechanics of systems with gauge symmetry.

The book reveals how gauge symmetry may lead to a non-trivial geometry of the physical phase space and studies its effect on quantum dynamics by path integral methods. It also covers aspects of Hamiltonian path integral formalism in detail, along with a number of related topics such as the theory of canonical transformations on phase space supermanifolds, non-commutativity of canonical quantization, and elimination of non-physical variables. The discussion is accompanied by numerous detailed examples of dynamical models with gauge symmetries, clearly illustrating the key concepts.

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Preface

In the past four decades, gauge theories have become dominant in high energy physics. All known fundamental interactions – strong, weak, electromagnetic, and gravitational, are described by gauge-invariant Lagrangians (actions). In the framework of classical physics, gauge invariance does not cause any significant theoretical problems. In the case of electrodynamics, which has always served as the benchmark for all other field theories, gauge arbitrariness can be eliminated by adding a suitable supplementary (gauge) condition on the vector potential, e.g. the Lorenz condition, to the equations of motion. In gravity, the De Donder–Fock gauge is often used. But initial studies on quantization of electromagnetic fields showed that gauge theories required a special approach. Canonical quantization demands the existence of the Hamiltonian formalism, whose construction turned out to be not an easy task. It appeared that in electrodynamics the equations that relate generalized velocities and canonical momenta cannot be solved for the former, i.e. the velocities cannot be expressed as functions of the generalized coordinates and momenta (a consequence of the Lagrangian being singular (or degenerate)). As a result, conditions on canonical variables (constraints) occur. It was required, first, to formulate the theory of dynamical systems with constraints and, second, to find a consistent procedure for their quantization.

For electrodynamics these problems had already been solved by W. Heisenberg and W. Pauli in 1930. In a further development of modern quantum electrodynamics (QED), it appeared to be possible to avoid the essential problem of quantization in the presence of constraints. It was established that in QED one can develop perturbation theory insensitive to the peculiarities of Hamiltonian dynamics associated with the degeneracy of the Lagrangian. The only difference with a non-gauge theory was the existence of the Ward identities, while the only additional concern was the necessity to watch for the gauge invariance of the results. However, attempts to describe more complicated constrained systems (such as, e.g. gravity) demanded the development of a general theory. This theory was developed by P. A. M. Dirac and G. P. Bergmann; Dirac gave the quantization procedure for such theories. For quite some time these results did not attract much attention from the experts. Appreciation came after the discovery of the unified theory of weak and electromagnetic interactions. The Yang–Mills fields (i.e. gauge theories with non-Abelian groups) were recognized as fundamental in nature. In the theory a small parameter still existed and, therefore, it seemed that nothing could prevent the application of perturbation theory.

However, it had earlier been established by R. Feynman that a straightforward extension of the standard QED formalism to the Yang–Mills theory would lead to catastrophe – the breaking of unitarity. This was a clear indication of the necessity for a general theory.

The situation became even more desperate after the discovery that strong interactions are also described by a non-Abelian gauge theory. Here the problem of a correct description was truly of paramount importance because the interaction could only be assumed to be small at short distances. The limitations of the trouble-free perturbation theory method used so far were emphasized by the phenomenon of quark confinement which, in particular, is characterized by a linearly rising potential. More to point, the very applicability of perturbation theory turned out to be questionable (because of the increase of the coupling constant with distance), the new theory of strong interactions, quantum chromodynamics (QCD), did not seem to provide even a qualitative explanation of confinement. The problem was made even worse by the fact that invoking new methods such as quasiclassical ones, not related to perturbation theory, was unsuccessful. Few results from the first ambush on the problem justified a subsequent siege.

It seems that some essential features of QCD escaped the attention of the physicists. The necessity to scrutinize the very foundations of the theory thus became evident and the starting point for such an analysis would be the fundamental works of Dirac and Bergmann.

This monograph, to a large extent, is devoted to studies of the foundations of the dynamics of gauge systems (constrained systems). As some of the most important, general features of gauge dynamics were not investigated thoroughly enough, this presentation is focused on the simplest models.[1] They are quite elementary, but at the same time, they possess all the characteristic features inherent in theories of this kind. Clearly, a good knowledge of simple gauge systems is very helpful for the understanding of gauge theories with infinitely many degrees of freedom, i.e. field theories. These models are Yang–Mills theories with matter fields in spacetime of the reduced dimension (0+1). Inspite of their maximal simplicity, but also maybe because of it, they have turned out to be quite useful. Studies of these models have led to the uncovering of a few peculiarities of gauge dynamics which were previously unnoticed. Even the simplest model (the scalar electrodynamics in spacetime (0+1)) allowed such a remarkable fact as physical phase space reduction to be revealed. It appears that the phase space of the only physical degree of freedom in this model is not a plane but rather a *cone* unfoldable into a half-plane. This drastically changes the dynamics of the system. For example, the frequency of the harmonic oscillator is doubled (consequently, the distance between its energy levels is also doubled in quantum theory). In models with an arbitrary gauge group, the structure of the physical

[1] It is said that, when introduced to a new phenomenon, W. K. Rentgen used to ask the question, “Where is the hydrogen atom here?”

phase space is determined by the Weyl group W, the discrete subgroup that characterizes the root pattern of the gauge group (W is the group of all compositions of the reflections in the hyperplanes orthogonal to simple roots). A remarkable synthesis of physics and rather abstract mathematics! Another example of the usefulness of studying simple gauge models is the gauge fixing problem. The insufficiency of the Lorenz gauge condition in eliminating gauge arbitrariness, discovered in non-Abelian gauge field theories, attracted much attention. Studies of this problem in simple models provide complete clarification of the issue, as well as a correct recipe for a quantum description in an arbitrary gauge.

The aforementioned examples already seem sufficient to indicate that the properties of gauge and non-gauge systems may be quite different, experience gained in working with ordinary systems might be insufficient, and without a good understanding of the mechanics of simple gauge systems, it would be difficult to figure out the peculiarities of gauge systems with many degrees of freedom. This is especially true for gauge field theories that have infinitely many degrees of freedom.

The years that have passed since the early studies on gauge theories have only shown that the importance of problems in the Hamiltonian formalism for gauge theories and the path integral method do not diminish. The results of both prominent and lesser known researchers scattered over the vast journal literature have not lost their significance for a new generation of theoretical physicists. The Hamiltonian mechanics resides in the very foundations of all physics. It seems more and more evident that the beauty of the world is based on chance. Namely, if one admits the existence of a fundamental source of random forces (a "thermostat"), then there naturally emerge the notions of the phase space, the Hamiltonian, Hamiltonian mechanics, quantum mechanics (the probability amplitudes, Planck constant, Schrödinger equation, and Fock space), the time arrow, and many other common features of our present physical understanding of nature. In this newly emerging picture, there is a place for the Nambu mechanics with several Hamiltonians. At the foundation of all this richness lies the Gibbs distribution which, in the simplest and most important case of the harmonic oscillator, is the consequence of the central limit theorem (the two-dimensional normal (Gauss) distribution). But the probability processes are described by the Wiener path integral. It turns out that the path integrals in quantum mechanics have the very same nature and emerge due to the very same reason. Recently a model of the "physical spacetime" has been proposed (first advocated by V. A. Fock). Its essence is that 3D space is modeled by a 3D network built of bosonic strings and placed into a thermostat. In this model there naturally emerge the Minkowski space, fermions, the idea of supersymmetry, and so on. At the same time, there appears a new aspect of Hamiltonian mechanics. The network is not a manifold, while Hamiltonian mechanics is formulated on an even-dimensional manifold. So there is a problem of developing the Hamiltonian mechanics on spaces which are not manifolds. This development is still very much in progress. Thus, it is evident that the problems of modern fundamental physics require a deep understanding and study of the mechanics of gauge systems.

Chapter 1 provides a survey of some classical aspects of Hamiltonian mechanics that are used throughout the book (Hamilton's least action principle, Hamiltonian equations of motion, canonical transformations, and so on). In addition to the traditional material, particular topics whose development was stimulated by the demands of modern quantum field theory are also included. First of all, one such topic is the mechanics with Grassmann (anticommuting) variables and the mechanics of mixed systems with bosonic and fermionic degrees of freedom. Noether's theorem is extended to Lagrangian systems on supermanifolds. It is illustrated with several examples including supersymmetry. A general formalism of the Hamiltonian mechanics on supermanifolds is completed with the Hamilton–Jacobi theory. Some extensions of Hamiltonian mechanics are discussed, e.g. Nambu mechanics. The question of non-canonical transformations, that are beginning to gain significance in modern physics, but whose general theory is still largely undeveloped, is also studied here. A classical example can be furnished by the holomorphic representation of the harmonic oscillator problem which is obtained by the transformation $(q, p) \to (a, a^*)$ which does not preserve the Poisson bracket. Another example is related to the so-called q-deformed systems (dynamical systems associated with quantum groups), the research area that became quite fashionable two decades ago. The last section of Chapter 1 contains a brief review of recent applications of Hamiltonian mechanics to the aforementioned problems in fundamental physics (the Gibbs distribution and the emergence of quantum mechanics). Overall, this chapter can be used as a modern supplement to a classical university course in theoretical mechanics.

Chapter 2 is devoted to the fundamentals of quantum theory, namely, those that concern the Hamiltonian path integral formalism. Although the basics of the formalism have already been established by Feynman, there are many aspects, important in practice, that have been studied much later. Among the most significant are the problem of changing variables and path integrals in curvilinear coordinates whose coordinate surfaces have a non-trivial topology, the behavior of the Hamiltonian path integral under canonical transformations, the operator ordering problem in the Hamiltonian, the path integral formalism for problems with boundaries, and others. The above listed problems are by no means far-fetched, they are encountered in the path integral formalism for the simplest constrained systems. The methods developed allow one to formulate the path integral formalism for problems with non-standard phase spaces. Many of the issues studied in this chapter are not usually included in standard textbooks on the path integral method.

Chapter 3 provides an elementary introduction to the theory of constrained systems. The presentation is illustrated by many examples. This chapter is not intended to give a comprehensive review of the vast subject of constrained dynamics. It merely offers the basic facts with an emphasis on what is relevant for the sequel.

Chapter 4 is devoted to the problem of quantization of constrained systems. In addition to the standard quantization recipes, the operator ordering problem in constraints is considered, and a particular non-standard case, when the "constraints" depend on velocities, but cannot be solved for these, is studied as well (Section 4.3).

In Chapter 5 dynamical systems with gauge symmetries are analyzed in detail. The physical phase space structure, the problem of eliminating non-physical degrees of freedom, and other peculiarities of the gauge dynamics are studied. Models with Abelian and non-Abelian gauge groups describing dynamics of both bosonic and fermionic degrees of freedom (mixed systems with commuting and anticommuting variables) are considered. The question of choosing a gauge fixing condition is thoroughly investigated as well as peculiarities of a gauge-fixed dynamics. In particular, with an example of Yang–Mills theories on cylindrical spacetime, such important issues as orbit space geometry, physical phase space structure, quantum theory on the orbit space, gauge-fixing ambiguity problem (Gribov problem), and their effects on a quantum description are presented. The emphasis is put on proving the equivalence of quantum theories where all non-physical degrees of freedom are eliminated (by, e.g. a gauge fixing) to explicitly gauge invariant quantum theories based on the Dirac quantization method. Some simple effects due to non-standard physical phase space structure are studied in gauge quantum theories. Most of the subject matter of this chapter can only be found in journal publications.

In Chapter 6, the Hamiltonian path integral formalism is developed for the models from Chapter 5. The main feature here is the non-standard structure of the physical phase space and its effects on the path integral formalism in gauge theories. The problem is tackled in two ways. First, the operator formalism for physical degrees of freedom with a non-standard phase space, developed in Chapter 5 for gauge theories, is applied to derive the corresponding path integral formalism by means of the methods of Chapter 2. In this way, the equivalence of the path integral formalism to the Dirac quantization method is established. Then a new method of developing the path integral formalism for gauge theories is presented. It is based on the projection method for constrained systems and no longer refers to the operator formalism. It offers a modification of the Kato–Trotter formula for an infinitesimal transition amplitude in the presence of a gauge symmetry. Its equivalence to the Dirac operator formalism is proved. Several examples are worked out in detail to demonstrate how the orbit space geometry and physical phase space structure are accounted for in the path integral formalism. The path integral formalism developed here solves the gauge fixing ambiguity problem. With an explicit example of Yang–Mills theories on the cylindrical spacetime, the modified path integral is shown to yield the correct spectrum of the theory that has been obtained by means of rigorous methods of axiomatic quantum field theory. A relation between the developed path integral formalism and the Morse theory used to study the orbit space structure in 4D Yang-Mills theory is established. The chapter is concluded with a study of the effects of a non-standard phase space of physical variables in gauge theories on their Green's functions. The material included in this chapter can only be found in the journal literature.

Chapter 7 is special. In the preceding chapters the simplest (soluble) gauge models are mostly studied, here the discussion concerns gauge field theories and, moreover, one of the most difficult problems in gauge field theories – confinement. This chapter is included in this monograph because its content is, in essence, a consistent implementation of the Dirac method developed for general constrained systems. An understanding of the key role of constraints and the extended group of gauge transformations allows one to approach the phenomenon from a somewhat more general point of view. The universal nature of the phenomenon becomes evident in this approach. It also makes it possible to separate features inherent only in QCD from those inherent in any gauge theory. From the viewpoint of fiber bundle theory, the path-ordered exponential (the operator of parallel transport) is the only basic building block in all gauge invariants. Coherent field excitations corresponding to the path-ordered exponential (under the assumption of their dynamical stability) are characterized by the energy rising linearly with the path length. In other words, objects with energies rising linearly with the distance between them are internally inherent to such theories and emerge at the very beginning of the quantum theory development. On the other hand, the relative simplicity in establishing the basic properties of confinement only emphasizes the real difficulties of the concrete problem of describing hadrons in QCD, even in the simplest limiting case of massive quarks and a large coupling constant. The problem of the dynamic stability of coherent excitations of quantum gauge fields generated by gauge invariant operators is illustrated with the soluble example of the decay of string-like excitations of quantum electromagnetic fields, followed by formation of the Coulomb field of static sources. This chapter also contains an analysis of gauge invariants in gauge theories with the Higgs phenomenon.

All supplementary material is included in Chapter 8. This chapter contains some basic facts on group theory, Lie algebras, Cartan–Weyl basis (which is used extensively in Chapters 5 and 6), a brief survey of Grassmann algebras and calculus on them, solving of equations of motion in dynamical systems on supermanifolds, standard gauge fixing procedures in the Hamiltonian path integral formalism, and some other technical facts that will be helpful when reading the book.

This book should be accessible to upper level undergraduate students and can certainly be used by graduate students specializing in theoretical and mathematical physics, although its content is based on recent studies which have not been reviewed in the monograph literature. Each of the chapters can be read independently. The introduction to each chapter is quite elementary and does not require special prerequisites beyond standard university courses on analytical mechanics and quantum theory. Exceptions to this may be Chapters 5, 6, and 7. For these the reader may need some basic knowledge of group theory and fiber bundle theory (although, the latter is not really necessary).

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