CAMBRIDGE

Cambridge University Press
978-0-521-89512-5 - Hamiltonian Mechanics of Gauge Systems
Lev V. Prokhorov and Sergei V. Shabanov
Excerpt
More information

1
Hamiltonian formalism

1.1 Hamilton's principle of stationary action

A dynamical system with a finite number n degrees of freedom can be described by real functions of time $q^i(t)$ $(i = 1, 2, ..., n)$ which, together with the derivatives $\dot{q}^i(t)$, uniquely determine its state at any moment of time t. The collection of all values of q^i is called the configuration space M of the system. In the simplest case, M is a Euclidean space R^n. It can generally be an n-dimensional (differentiable) manifold. In a small neighborhood of each point, a manifold looks topologically like a Euclidean space R^n. This allows one to introduce local coordinates on the manifold M and develop calculus on M. Every point of M has a neighborhood U that is homeomorphic to R^n (there is a continuous and invertible map between points of U and some subset of R^n). Let q^i, $i = 1, 2, ..., n$, be rectangular coordinates in R^n. A neighborhood U with local coordinates in it (i.e. with the map $\phi : \ U \to R^n$) is called a chart. A manifold can be covered by a set of charts with transition maps between local coordinate systems in the overlap of any two charts. Thus, in a local coordinate system on M, the time evolution of a dynamical system is again described by n real functions $q^i(t)$, $i = 1, 2, ..., n$.

Consider trajectories of the system $q^i = q^i(t)$ on M through a point $q^i_0 = q^i(0)$. The derivative $\dot{q}^i(0)$ is called a tangent vector to the curve at $q = q_0$. The tangent space T_q at $q = q_0$ is defined as the set of all tangent vectors $\dot{q}^i(0)$ at the point $q = q_0$. Thus, the velocity $\dot{q}$ is an element of T_q. The tangent bundle of a manifold M is a disjoint union $TM = \cup_{q\in M} T_q$. A state of a dynamical system is an element of TM as, at any moment of time t, it is defined by values of $q^i(t)$ and $\dot{q}^i(t)$. The quantities q^i and $\dot{q}^i$ are called generalized coordinates and velocities, respectively. It is worth noting that a configuration space can be infinite-dimensional if the system has infinitely many degrees of freedom. A field theory is an example of such a system.

Differential equations whose solution determines the trajectory of motion $q = q(t)$ in the system configuration space are called the equations of motion. For a sufficiently large class of dynamical systems, the equations of motion can be obtained from *Hamilton's principle of staionary action* (or, simply, Hamilton's principle) which states that the trajectory $q(t)$ with fixed end points $q(t_1)$ and $q(t_2)$ is an extremum of the action functional (see, e.g. [1–3])

$$S = S[q; t_1, t_2] = \int_{t_1}^{t_2} L(\dot{q}, q, t)\, dt. \tag{1.1}$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-89512-5 - Hamiltonian Mechanics of Gauge Systems
Lev V. Prokhorov and Sergei V. Shabanov
Excerpt
More information

The real function L of time $t \in R$, generalized coordinates, and velocities is called a Lagrangian of the dynamical system (i.e. $L: \ R \times TM \to R$).

Variations of a trajectory $q^i = q^i(t)$, $t_1 \le t \le t_2$, are a set of trajectories $q^i = q^i_\epsilon(t)$ parameterized by real ϵ so that

$$q^i_\epsilon(t) = q^i(t) + \epsilon\Phi^i(t) + O(\epsilon^2) \equiv q^i(t) + \delta q^i(t) + O(\epsilon^2), \tag{1.2}$$

and $\Phi^i(t_{1,2}) = 0$, where $\Phi^i(t)$ is called the variation field. The variational derivative $\delta S/\delta q$ is defined by

$$\frac{d}{d\epsilon} S[q_\epsilon; t_1, t_2]\Big|_{\epsilon=0} = \int_{t_1}^{t_2} \frac{\delta S}{\delta q^i}\ \Phi^i(t)\, dt. \tag{1.3}$$

A curve $q^i = q^i(t)$ is an extremum of the action (1.1) if the variational derivative of the latter vanishes for this curve. The extremum condition

$$\frac{\delta S}{\delta q^i(t)} = 0\ , \qquad \delta q(t_1) = \delta q(t_2) = 0, \tag{1.4}$$

yields the Euler–Lagrange equation of motion

$$\frac{d}{dt}\frac{\partial}{\partial \dot q^i}\ L(\dot q, q, t) = \frac{\partial}{\partial q^i}\ L(\dot q, q, t). \tag{1.5}$$

Thus, all such dynamical systems are determined by their Lagrangians. A Lagrangian should satisfy some physical conditions, e.g. to be invariant under symmetry transformations of the dynamical system such as, for instance, the Lorentz symmetry transformations, etc. It is natural to demand that the Euler–Lagrange equations be self-consistent. For example, put $L = q$ (here $n = 1$). Equation (1.5) leads to an obvious contradiction, $0 = 1$. In other words, the corresponding action has no extremum at all.

It appears that most dynamical systems occurring in nature are described by second-order differential equations, that is, they do not involve higher-order derivatives with respect to time. However, there is no general law that prohibits theories with higher derivatives. As a rule, the existence of higher derivatives in the theory is associated with some internal structure of the object described by such a theory. For example, higher derivatives naturally emerge in an approximate description of elastic vibrations of a rod whose transverse dimensions are much smaller than its length [4]. In this case, the Lagrangian depends on higher time derivatives $q^{i(m)} = d^m q^i/dt^m$ $(m > 1)$ so that the trajectory of motion is an extremum of the action

$$S = S[q; t_1, t_2] = \int_{t_1}^{t_2} L\left(q^{(m)}, q^{(m-1)}, ..., \dot q, q, t\right)\, dt.$$

The equations of motion resulting from the principle of stationary action (called the Hamilton–Ostrogradsky principle) are known as Ostrogradsky's equations [5]

$$\frac{\delta S}{\delta q^i(t)} = \sum_{k=0}^{m} (-1)^k\, \frac{d^k}{dt^k}\, \frac{\partial L}{\partial q^{i(k)}} = 0. \tag{1.6}$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-89512-5 - Hamiltonian Mechanics of Gauge Systems
Lev V. Prokhorov and Sergei V. Shabanov
Excerpt
More information

Note that the variational derivative of the action is evaluated under the condition that $\delta q^{(k)}(t_1) = \delta q^{(k)}(t_2) = 0$, for $k = 0, 1, ..., m-1$.

1.1.1 Poincaré equations

Hamilton's principle on manifolds has a peculiarity related to the choice of a basis in the tangent space. At any point $q \in M$ the operators (vector fields) $\hat{e}_i = \partial/\partial q^i$ define the rate of change of any function $F(q)$ along the trajectory $q^i = q^i(t)$: $\dot{F} = \dot{q}^i \hat{e}_i F$. The velocity $\dot{q}^i$ is an element of the tangent space T_q. In general, one can define n smooth linearly independent vector fields on M that serve as a new basis in T_q,

$$\hat{w}_i = w_i^j(q)\hat{e}_j, \qquad [\hat{w}_i, \hat{w}_j] = c_{ij}^k(q)\hat{w}_k, \tag{1.7}$$

where $w_i^j(q)$ are functions of q and $[\,,\,]$ denotes the commutator. The rate of change of a function F along the trajectory can be written in the new basis:

$$\dot{F} = \dot{q}^i \hat{e}_i F = \omega^i \hat{w}_i F.$$

The quantities ω^i are position-dependent linear combinations of the velocities $\dot{q}^i$ and called *quasi-velocities* [6]. In differential geometry, the structure functions $c_{ij}^k(q)$ are called the *object of anholonomity* [7]. Note that under a change of local coordinates on M, $q \to Q$, the basis in T_q transforms as $\partial/\partial Q^i = J_i^j(q)\hat{e}_j$, where $J_i^j = \partial q^j/\partial Q^i$ is the Jacobian of the change of coordinates. Clearly, the new basis $\hat{w}_i = \partial/\partial Q^i$ is commutative, i.e. if $w_i^j = J_i^j$, then the structure functions vanish, $c_{ij}^k = 0$. When the structure functions are non-trivial, the basis vector fields are not integrable (there is no change of variables such that $\hat{w}_i = \partial/\partial Q^i$), and, hence, the quasi-velocity ω^i is not the time derivative of a local coordinate on M. Such bases in T_q are called *anholonomic*. Anholonomic bases turn out to be quite useful in constructing equations of motion for dynamical systems on manifolds.

Let $L(\dot{q}, q) = \tilde{L}(\omega, q)$ be the Lagrangian expressed as a function of local coordinates and quasi-velocities. Let $q_\epsilon^i(t)$ be a variation of the trajectory as defined in (1.2). Put

$$\frac{\partial}{\partial \epsilon} F(q_\epsilon)\Big|_{\epsilon=0} = \phi^i \hat{w}_i F, \tag{1.8}$$

where ϕ^i are the components of the variation vector field Φ in the new basis. They are now independent variations of the trajectory. From the commutativity of the derivatives, $[\partial/\partial t, \partial/\partial\epsilon]F(q_\epsilon(t)) = 0$, and the commutation relation (1.7), the variations of the quasi-velocities are obtained:

$$\frac{\partial \omega^i}{\partial \epsilon}\Big|_{\epsilon=0} = \frac{d\phi^i}{dt} + c_{kj}^i \omega^k \phi^j. \tag{1.9}$$

Computing the variation of the action $S = \int \tilde{L} dt$ by means of (1.8), (1.9), and the boundary conditions $\phi^i(t_{1,2}) = 0$ (as a consequence of $\Phi^i(t_{1,2}) = 0$), the

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-89512-5 - Hamiltonian Mechanics of Gauge Systems
Lev V. Prokhorov and Sergei V. Shabanov
Excerpt
More information

equations of motion in the anholonomic basis are obtained:

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \omega^i} = \hat{w}_i \tilde{L} + c^j_{ik}\omega^k \frac{\partial \tilde{L}}{\partial \omega^j}. \tag{1.10}$$

Equations (1.10) are called the Poincaré equations [8]. Poincaré first obtained them in his studies of celestial mechanics. In a holonomic basis in T_q the Poincaré equations become the Euler–Lagrange equation (1.5).

To appreciate the Poincaré equations, consider a dynamics on a group manifold $M = G$. Let $\hat{w}_i$ be generators of left shifts on G, i.e. $g \to g_0 g$, $g_0, g \in G$ (see Section 8.1.1 for a summary of the group theory). Then the structure functions $c^k_{ij}(q)$ are constant and coincide with the structure constants of G. If, in addition, one imposes the condition that the Lagrangian is invariant under the left shifts, i.e. $\hat{w}_i \tilde{L} = 0$, then the Poincaré equations determine the time evolution of a "free" particle on the group manifold (in the form that does not depend on any particular choice of local coordinates on G) [6].

The simplest example of this kind is provided by the motion of a rigid body with one fixed point. The motion has the characteristic properties that the distance between any two points of the body remains constant, and one of its points always coincides with a fixed point in space. Clearly, all possible positions of the system are obtained from a particular one by rotations in space about a point. Thus, M is the group $SO(3)$ manifold. Let $\mathbf{r}_0$ be the position vector of a point of the body relative to the point about which it rotates. In time t the position vector is $\mathbf{r}(t) = U(t)\mathbf{r}_0$, where $U(t) \in SO(3)$ (i.e. $UU^T = U^T U = 1$ and $\det U = 1$, where U^T is the transpose of U). The velocity of this point is then $\dot{\mathbf{r}} = \dot{U}U^T\mathbf{r}$. Since the matrix $\dot{U}U^T$ is skew-symmetric (because $UU^T = 1$), there exists a vector $\boldsymbol{\Omega}$, called the angular velocity, such that $\dot{\mathbf{r}} = \boldsymbol{\Omega} \times \mathbf{r}$. In the frame moving with the body, the angular velocity vector is $\boldsymbol{\omega} = U^T\boldsymbol{\Omega}$. It can be shown that the kinetic energy of the system has the form (see, e.g. [1])

$$L = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right),$$

where the components ω_i are defined in the frame whose axes coincide with the principal axes of the tensor of momenta of inertia, and I_i $(i = 1, 2, 3)$ are principal values of the tensor. Apparently, ω_i are not the conventional generalized velocities of the dynamical system as there exist no coordinates q_i such that $\omega_i = \dot{q}_i$. So the Euler–Lagrange equation (1.5) would lead to incorrect equations for the rigid body dynamics, and the Poincaré equations must be applied. In the basis associated with the principal axes, $\hat{w}_i$ should generate rotations about the ith axis, and, hence, $[\hat{w}_i, \hat{w}_j] = \varepsilon_{ijk}\hat{w}_k$, where ε_{ijk} is the totally skew-symmetric unit tensor (the structure constants of $SO(3)$). According to (1.10), one obtains

$$\dot{\mathbf{L}} = \mathbf{L} \times \boldsymbol{\omega},$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-89512-5 - Hamiltonian Mechanics of Gauge Systems
Lev V. Prokhorov and Sergei V. Shabanov
Excerpt
More information

where $L_i = I_i\omega_i$ (no summation over i) are the components of the angular momentum vector $\mathbf{L}$. These are the well-known equations of a spinning top.

1.1.2 The existence of a Lagrangian for a dynamical system

Suppose that the time evolution of a dynamical system is described by a system of second-order differential equations. A natural question arises as to whether this system can always be cast in the form of the Euler–Lagrange equations. The answer is negative. There are so-called *non-Lagrangian* systems for which it is impossible to find a Lagrangian [9]. As an example, consider a dynamical system in $M = R^3$ whose time evolution is determined by the equations [10, 11]

$$\ddot{q}_i - \alpha\epsilon_{ijk}q^j\dot{q}^k = 0, \tag{1.11}$$

where α is a real constant, ϵ_{ijk} is the totally antisymmetric unit tensor, $\epsilon_{123} = 1$ (in [10], a more general system was studied where $\alpha = \alpha(q) = \lambda|\mathbf{q}|^{-3}$, $\lambda =$ const). The summation over repeated indices is assumed in (1.11). Clearly, the initial value problem for Eqs. (1.11) does have a solution. However, there is no Lagrangian such that the Euler–Lagrange equations coincide with (1.11).

In general, the existence of a Lagrangian for a dynamical system was studied by Helmholtz [12]. Given a system of second-order equations

$$G_i(\ddot{q}, \dot{q}, q) = H_{ij}(\dot{q}, q)\ddot{q}^j + A_i(\dot{q}, q) = 0,$$

where $i = 1, 2, ..., n$, the necessary and sufficient conditions on G_i as functions of q, $\dot{q}$, and $\ddot{q}$ in order for a Lagrangian L to exist are [12]

$$\frac{\partial G_i}{\partial \ddot{q}^j} = \frac{\partial G_j}{\partial \ddot{q}^i}, \tag{1.12}$$

$$\frac{\partial G_i}{\partial \dot{q}^j} + \frac{\partial G_j}{\partial \dot{q}^i} = \frac{d}{dt}\left(\frac{\partial G_i}{\partial \ddot{q}^j} + \frac{\partial G_j}{\partial \ddot{q}^i}\right), \tag{1.13}$$

$$\frac{\partial G_i}{\partial q^j} - \frac{\partial G_j}{\partial q^i} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial G_i}{\partial \dot{q}^j} - \frac{\partial G_j}{\partial \dot{q}^i}\right). \tag{1.14}$$

If these conditions are fulfilled, a Lagrangian L can be found by solving the following equations

$$\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j - \frac{\partial L}{\partial q^i} = G_i. \tag{1.15}$$

It is easy to verify that for G_i given in (1.11), conditions (1.12) and (1.13) are trivially satisfied, while the third Helmholtz condition (1.14) is violated.

However, the existence problem for a Lagrangian may be treated more broadly. Namely, one could take an *equivalent* set of equations $M_i^j G_j = 0$, where *integrating factors* M_i^j form a non-singular matrix and are functions of q, $\dot{q}$, and t. Then the Helmholtz conditions are viewed as equations for M_i^j. These equations can be recast in a geometrical form known as the Douglas theorem [13] (see also

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-89512-5 - Hamiltonian Mechanics of Gauge Systems
Lev V. Prokhorov and Sergei V. Shabanov
Excerpt
More information

Section 1.14.3). In particular, it should be noted that there are dynamical systems for which there exist many integrating factors leading to different Lagrangians.

As a simple example, consider a particle moving along the line under the friction force proportional to the particle velocity. The equation of motion has the form

$$\ddot{q} + \alpha\dot{q} = 0. \tag{1.16}$$

The Helmholtz condition (1.13) is not fulfilled for this equation. Consider an equivalent equation of the form

$$G \equiv M(\dot{q}, q)(\ddot{q} + \alpha\dot{q}) = 0.$$

The integrating factor M must be chosen so that G satisfies (1.13) (the other Helmholtz conditions are trivially satisfied for a one-dimensional motion):

$$\alpha\dot{q}\,\frac{\partial M}{\partial \dot{q}} + \alpha M = \dot{q}\,\frac{\partial M}{\partial q}. \tag{1.17}$$

This equation has many solutions and can easily be solved by separating variables $M(\dot{q}, q) = \Phi(\dot{q})\Psi(q)$. The corresponding Lagrangian is obtained by solving Eq. (1.15). Let $\beta = -\Psi'/\Psi \neq 0$ be the separation constant. Then $M(\dot{q}, q) = e^{-\beta q - \beta\dot{q}/\alpha_{/\dot{q}}}$. Non-constant solutions ($\dot{q} \neq 0$) of (1.16) are extrema of the action

$$S = \int dtL, \qquad L = \dot{q}F(\dot{q})\exp(-\beta q), \tag{1.18}$$

where the function $F(\dot{q})$ satisfies the condition

$$\frac{dF}{d\dot{q}} = \frac{1}{\dot{q}^2}\exp\left(-\frac{\beta}{\alpha}\dot{q}\right).$$

If the separation constant in (1.17) is set to zero, i.e. $M = \Phi(\dot{q}) = 1/\dot{q}$ ($\beta = 0$), then the Lagrangian has the conventional form $L = T(\dot{q}) - V(q)$, where $V = \alpha q$ and $d^2T/d\dot{q}^2 = 1/\dot{q}$.

Alternatively, the equation of motion (1.16) can also be regarded as the Euler–Lagrange equation for the Lagrangian that explicitly depends on time:

$$L = \frac{1}{2}e^{\alpha t}\dot{q}^2. \tag{1.19}$$

The Helmholtz condition (1.13) is satisfied with the integrating factor explicitly depending on time, $M = e^{\alpha t}$. Furthermore, the dynamical system (1.16) can be viewed as a part of a larger Lagrangian dynamical system, e.g.

$$L(\dot{q}, q, \dot{Q}, Q) = \dot{Q}(\dot{q} + \alpha q).$$

The Euler–Lagrange equations of motion for the variables q and Q are decoupled, and, in particular, they coincide with (1.16) for $q(t)$. Such a possibility is not even included in the Helmholtz conditions.

The above analysis shows that the choice of the Lagrangian may not be unique, and additional physical principles should be invoked to limit it (e.g. to demand

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-89512-5 - Hamiltonian Mechanics of Gauge Systems
Lev V. Prokhorov and Sergei V. Shabanov
Excerpt
More information

that the Hamiltonian coincides with the system energy (see Section 1.2), or that the Lagrangian has specific symmetries, etc.).

1.2 Hamiltonian equations of motion

Equations (1.5) are of the second order. Any system of second-order differential equations can be transformed to a system of first-order differential equations by increasing the number of independent functions. Indeed, setting

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \tag{1.20}$$

one finds from (1.5) that

$$\dot{p}_i = \frac{\partial L}{\partial q^i}. \tag{1.21}$$

The quantity p_i defined by (1.20) is called the canonical momentum conjugated to q^i. The matrix

$$T_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$$

is called the *Hessian matrix* (or simply, the *Hessian*) of the Lagrangian system. If it is not singular, $\det T \neq 0$, then by the implicit function theorem, the relation (1.20) defines the generalized velocities $\dot{q}^i$ as functions of q^i and p_i, that is,

$$\dot{q}^i = \dot{q}^i(p, q, t), \tag{1.22}$$

which must be substituted into (1.21). Then Eqs. (1.21) and (1.22) comprise the system of first-order differential equations for the generalized coordinates q^i and momenta p_i.

In Lagrangian systems with a singular Hessian matrix, the implicit function theorem does not apply, and the straightforward transformation to the corresponding first-order equations of motion is impossible. In this case, Eqs. (1.20) yield some relations $\phi_a(p, q, t) = 0$, called *constraints*, between the canonical variables. A general analysis of *constrained* systems will be given in Chapter 3. Here it is always assumed that the Hessian matrix is not singular.

Let f be a concave function in a Euclidean space. The matrix of its second derivatives, $\partial^2 f/\partial x^i \partial x^j$, is either positive or negative definite in the domain of f. By the implicit function theorem, the equation $\mathbf{y} = \partial f/\partial \mathbf{x}$ can be solved defining the function $\mathbf{x} = \mathbf{x}(\mathbf{y})$. The function

$$g(\mathbf{y}) = \mathbf{y} \cdot \mathbf{x}(\mathbf{y}) - f(\mathbf{x}(\mathbf{y}))$$

is called the Legendre transform of f [3].

The Hamiltonian of a Lagrangian system is the Legendre transform of $L(\dot{q}, q, t)$ with respect to the variable $\dot{q}$,

$$H(p, q, t) = p_i \dot{q}^i - L(\dot{q}, q, t), \tag{1.23}$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-89512-5 - Hamiltonian Mechanics of Gauge Systems
Lev V. Prokhorov and Sergei V. Shabanov
Excerpt
More information

where $\dot{q}^i = \dot{q}^i(p,q,t)$ is defined by (1.20) (i.e. $y_i = p_i$ and $x^i = \dot{q}^i$). The first-order differential equations (1.20) and (1.21) can also be obtained from the variational principle for the action written in the Hamiltonian form,

$$S_H = \int_{t_1}^{t_2} dt \left(p_i \dot{q}^i - H(p,q,t)\right), \tag{1.24}$$

with the same boundary conditions (1.4). It follows from (1.23) that the actions (1.24) and (1.1) are equivalent. In the Hamiltonian action (1.24), the independent variables are p_i and q^i. Therefore

$$\frac{\delta S_H}{\delta q^i} = -\dot{p}_i - \frac{\partial H}{\partial q^i} = 0, \qquad \frac{\delta S_H}{\delta p^i} = \dot{q}_i - \frac{\partial H}{\partial p^i} = 0,$$

or, after moving all the partial derivatives of the Hamiltonian to the right-hand side,

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{1.25}$$

The system (1.25) is called the *Hamiltonian equations of motion.*

Thus, if $q^i(t)$ satisfies the Euler–Lagrange equations, then the pairs $(p_i(t), q^i(t))$ solve the Hamiltonian equations of motion and vice versa, meaning that the Euler–Lagrange and Hamiltonian systems of equations are equivalent. All the pairs (p_i, q^i) form the *phase space* of the system. A typical Lagrangian considered in what follows reads

$$L = \frac{1}{2} g_{ij}(q)\dot{q}^i\dot{q}^j + A_i(q)\dot{q}^i - V(q);$$

here $V(q)$ is a potential energy, the matrix g_{ij} is assumed to be non-singular and may depend on the coordinates. The Euler–Lagrange equations of this system are easy to obtain

$$g_{ij}\ddot{q}^j = -\Gamma_{nj,i}\dot{q}^n\dot{q}^j + F_{ij}\dot{q}^j - V_{,i}, \tag{1.26}$$

where $F_{ij} = A_{j,i} - A_{i,j}$, the index after the comma denotes the partial derivative with respect to the corresponding coordinate, for instance, $g_{nj,i} \equiv \partial g_{nj}/\partial q^i$, and $\Gamma_{nj,i} = 1/2(g_{ni,j} + g_{ji,n} - g_{nj,i}) \equiv [nj,i]$ are the Christoffel symbols.

Consider the Hamiltonian formalism for this system. By the definition (1.20) the canonical momenta are

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = g_{ij}\dot{q}^j + A_i. \tag{1.27}$$

The generalized velocities are found as functions of the coordinates and momenta from (1.27),

$$\dot{q}^j = \dot{q}^j(p,q) = g^{ji}(q)(p_i - A_i(q)),$$

where g^{ij} is the inverse of the matrix g_{ij}, $g^{ij}g_{jk} = \delta^i_k$. The substitution of these relations into (1.23) yields the Hamiltonian of the system:

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-89512-5 - Hamiltonian Mechanics of Gauge Systems
Lev V. Prokhorov and Sergei V. Shabanov
Excerpt
More information

$$H = \frac{1}{2} g^{ij}(p_i - A_i)(p_j - A_j) + V. \tag{1.28}$$

It is straightforward to verify that, when $g_{ij} = \delta_{ij}$ and $A_i = 0$, this Hamiltonian turns into the sum of the kinetic and potential energies of a particle with unit mass whose position is given by the coordinates q^i.

Such a simple relation between the Hamiltonian and the system energy is not always possible. For example, a particle moving along an axis under the friction force is described by Eq. (1.16). If the Lagrangian is taken in the form (1.18), then the Hamiltonian obtained via the Legendre transform does not depend on time explicitly, and, therefore, is a conservative quantity. The Legendre transform for the time-dependent Lagrangian (1.19) leads to the Hamiltonian that also explicitly depends on time,

$$H = \frac{1}{2} e^{-\alpha t} p^2, \tag{1.29}$$

where $p = \partial L/\partial \dot{q} = e^{\alpha t}\dot{q}$. It follows from the Hamiltonian equations of motion (1.25) that $\dot{p} = 0$, i.e. the momentum is conserved, $p = p_0 = \text{const}$. Therefore,

$$H = e^{-\alpha t} E_0, \tag{1.30}$$

where $E_0 = p_0^2/2$ is the initial energy. Equation (1.30) shows that the Hamiltonian of the system is not conserved.

So, the same equations of motion can be obtained from different Lagrangians; the Hamiltonian depends on the choice of the Lagrangian. In Section 1.12 a particle with friction is also described as a system with a non-standard symplectic structure.

In addition, it is worth noting that there are dynamical systems for which the Hamiltonian equations of motion are impossible to construct. For example, put

$$p_i = \dot{q}_i, \qquad \dot{p}_i = \alpha \epsilon_{ijk} q_j p_k, \qquad i, j, k = 1, 2, 3. \tag{1.31}$$

Equations (1.31) are equivalent to the equations of motion (1.11) for which there exists no Lagrangian (see also a further discussion in Section 1.14.3).

1.3 The Poisson bracket

Consider the time evolution of a function $F(p, q, t)$ along a phase space trajectory $p = p(t)$ and $q = q(t)$. From the Hamiltonian equations of motion (1.25) one finds that

$$\begin{aligned} \frac{dF}{dt} &= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q^i}\dot{q}^i + \frac{\partial F}{\partial p_i}\dot{p}_i \\ &= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i} \equiv \frac{\partial F}{\partial t} + \{F, H\}. \end{aligned} \tag{1.32}$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-89512-5 - Hamiltonian Mechanics of Gauge Systems
Lev V. Prokhorov and Sergei V. Shabanov
Excerpt
More information

The symbol

$$\{A, B\} = \sum_{i=1}^{n} \left(\frac{\partial A}{\partial q^i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q^i} \right) \tag{1.33}$$

defined in (1.32) is called the *Poisson bracket* for functions A and B in the phase space of the system. The Poisson bracket has several remarkable properties. Namely, it is skew-symmetric,

$$\{A, B\} = -\{B, A\}, \tag{1.34}$$

satisfies both the Leibniz rule,

$$\{A, BC\} = \{A, B\}C + B\{A, C\}, \tag{1.35}$$

and the Jacobi identity,

$$\{\{A, B\}, C\} + \{\{B, C\}, A\} + \{\{C, A\}, B\} = 0 \tag{1.36}$$

for arbitrary A, B, and C. If the function F does not depend explicitly on time, then its evolution is determined by the equation $\dot{F} = \{F, H\}$. Taking the canonical coordinates and momenta as F, the Hamiltonian equations of motion are written in the symmetric form,

$$\dot{q}^i = \{q^i, H\}, \quad \dot{p}_i = \{p_i, H\}.$$

A quantity F is an *integral of motion* if $dF/dt = 0$ and, hence,

$$\frac{\partial F}{\partial t} + \{F, H\} = 0. \tag{1.37}$$

For integrals of motion that do not depend explicitly on time, Eq. (1.37) has a simpler form: $\{F, H\} = 0$. In particular, the Hamiltonian is an integral of motion if it does not depend explicitly on time because $dH/dt = \partial H/\partial t = 0$.

1.4 Canonical transformations

The Poisson bracket for the canonical coordinates and momenta is

$$\{q^i, p_j\} = \delta^i_j. \tag{1.38}$$

Consider functions

$$Q^i = Q^i(p, q) \ , \quad P_i = P_i(p, q) \tag{1.39}$$

such that

$$\{Q^i, P_j\} = \delta^i_j. \tag{1.40}$$

The functions (1.39) are said to define a *canonical transformation*, whereas Q^i and P_i are new generalized coordinates and momenta, respectively [1]. Put in (1.33),

© in this web service Cambridge University Press www.cambridge.org