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177 A Higher-Dimensional Sieve Method

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With Procedures for Computing Sieve Functions

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In memory of Hans-Egon (Ted) Richert

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Preface

Nearly a hundred years have passed since Viggo Brun invented his famous sieve, and yet the use of sieve methods is still evolving. At one time it seemed that, as analytic tools improved, the use of sieves would decline, and only their role as an auxiliary device would survive. However, as probability and combinatorics have penetrated the fabric of mathematical activity, so have sieve methods become more versatile and sophisticated, especially in conjunction with other theories and methods, until, in recent years, they have played a part in some spectacular achievements that herald new directions in mathematical discovery.

An account of all the exciting and diverse applications of sieve ideas, past and present, has yet to be written. In this monograph our aim is modest and narrowly focused: we construct (in Chapter 9) a hybrid of the Selberg [Sel47] and Rosser–Iwaniec [Iwa80] sieve methods to deal with problems of sieve dimension (or density) that are integers or half integers. This theory achieves somewhat sharper estimates than either of its ancestors, the former as given by Ankeny and Onishi [AO65]. The sort of application we have in mind is to show that a given polynomial with integer coefficients (some obvious cases excluded) assumes at integers or at primes infinitely many almost-prime values, that is, values that have few prime factors relative to the degree of the polynomial. To describe our procedure a little more precisely, we extend the pioneering method of Jurkat and Richert [JR65] for dimension 1 (that combined the Selberg sieve method with infinitely many iterations of the Buchstab identity) to higher dimensions by means of the Rosser–Iwaniec approach; in the process we give an alternative account of that approach.

The restriction we make to integer and half integer dimensions simplifies the analytic component of our method; an account avoiding this constraint exists [DHR88]–[DHR96], but is much more complicated. A justification for our restriction is that most sieve applications of the above kind occur in this context. We include an account of the case of dimension 1 because it serves as a model for what is to come and involves little extra work. While our treatment of that special case is not quite as sharp as in the classical exposition of Iwaniec [Iwa80] or that given more recently by Greaves [Grv01], it is somewhat simpler.

It should be said that our results for higher dimensions, unlike the case of dimension 1, are almost certainly not best possible, not even in a single instance; and that our approach might not be the right one there. Nevertheless, our method does have good applications, is simple to use, and, despite some complications of detail, rests solely on elementary combinatorial inequalities and relatively simple analysis. The combinatorics we have developed may in due course find other applications.

The first comprehensive account of sieve methods, by the second author and H.-E. Richert [HR74], appeared in 1974 and has been long out of print. Although it is also out of date in some important respects, we have tended to follow its overall design, and we have drawn on it for examples and applications.

We are happy to express our thanks to the many who have contributed to this work: the aforementioned authors, on whose ideas we have built; H.-E. Richert, who shared in our discoveries; our former students Ferrell S. Wheeler and David M. Bradley for their extensive computational assistance; our patrons, the University of Illinois and the National Science Foundation, who supported our research; our colleague A. J. Hildebrand for LaTeX and mathematical advice; Sidney Graham and Craig Franze for help in rooting out errors; and Cherri Davison, who skillfully and cheerfully converted our manuscript into LaTeX. Also, we thank our wives for their support during the preparation of this book.

The *Mathematica*®† package of sieve-related functions described in Appendix 1, as well as a list of comments and corrigenda, will be maintained at `http://www.math.uiuc.edu/SieveTheoryBook`. Finally, we request that readers advise us of any errors or obscurities they find. Our e-mail address is `sievetheorybook@math.uiuc.edu`.

† *Mathematica* is a registered trademark of Wolfram Research, Inc.

Notation

Standard terminology

$[x]$ denotes the largest integer not exceeding x.

$a \mid b$ means a divides b evenly, i.e., $b \equiv 0 \mod a$.

(a, b) denotes the greatest common divisor of the integers a and b (when no confusion with notation for an open interval is possible) and $\{a, b\}$ their least common multiple (see p. 14).

The symbols for the classical arithmetic functions have their usual meaning: $\mu(\cdot)$ is the Moebius function, $\tau(\cdot)$ the divisor function, $\phi(\cdot)$ Euler's totient function, $\pi(x)$ the number of primes not exceeding x, and $\pi(x, k, \ell)$ the number of primes not exceeding x and congruent to ℓ modulo k.

We use $\nu(\cdot)$ for the number of distinct prime divisors and $\Omega(\cdot)$ for the number of prime divisors counted according to multiplicity. Throughout Part I of this manuscript, $p(\cdot)$ and $p^+(\cdot)$ are the least and largest prime factors respectively of an integer (see p. 19).

The constants π and e have their usual meanings, and γ is always Euler's constant.

$O(\cdot)$ and $o(\cdot)$ have their usual meanings relating to the size of a function, and $O_z(\cdot)$ indicates dependence of the implied constant upon z.

$\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ denote integer sequences or sets, and $|\mathcal{A}|, |\mathcal{B}|, |\mathcal{C}|, \ldots$ their cardinalities; $\mathcal{A}_d$ denotes the sequence of multiples of d in $\mathcal{A}$. That is, $\mathcal{A}_d := \{a \in \mathcal{A} : d \mid a\} = \{a \in \mathcal{A} : a \equiv 0 \mod d\}$.

$\mathcal{P}$ is always a set of primes, the variable p denotes a prime throughout Part I of this book, and $\mathcal{P}^c$ is the set of primes not in $\mathcal{P}$.

$\mathbb{N}$ is the sequence of natural numbers, $\mathbb{Q}$ the set of rationals.

Sieve notation

The following lists indicate where sieve functions and sieve terminology are introduced and defined:

The notions of a function being *divisor closed* and/or *combinatorial* are defined on p. 27.

P_r denotes an integer having at most r prime factors, counted according to multiplicity; thus n is a P_r if $\Omega(n) \le r$ (see p. 141).

Multiplicative functions

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g	p. 15
g^*	p. 44
ρ	pp. 37, 39, 64, 125

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$R_{\mathcal{A}}(Y, z)$	p. 109 (9.27)

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For cases $\kappa > 1$, see also

$\chi^{\pm}(\cdot),\ \eta^{\pm}(\cdot)$	pp. 103–104 (9.5)–(9.7)
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