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176 The Monster Group and Majorana Involutions

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To Love and Nina

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Preface

The *Monster* is the most amazing among the finite simple groups. The best way to approach it is via an amalgam called the *Monster amalgam*.

Traditionally one of the following three strategies are used in order to construct a finite simple group H:

(I) realize H as the automorphism group of an object Ξ;
(II) define H in terms of generators and relations;
(III) identify H as a subgroup in a 'familiar' group F generated by given elements.

The strategy offered by the *amalgam method* is a symbiosis of the above three. Here the starting point is a carefully chosen generating system $\mathcal{H} = \{H_i \mid i \in I\}$ of subgroups in H. This system is being axiomatized under the name of *amalgam* and for a while lives a life of its own independently of H. In a sense this is almost like (III) although there is no 'global' group F (familiar or non-familiar) in which the generation takes place. Instead one considers the class of all *completions* of $\mathcal{H}$ which are groups containing a quotient of $\mathcal{H}$ as a generating set. The axioms of $\mathcal{H}$ as an abstract amalgam do not guarantee the existence of a completion which contains an isomorphic copy of $\mathcal{H}$. This is a familiar feature of (II): given generators and relations it is impossible to say in general whether the defined group is trivial or not. This analogy goes further through the *universal completion* whose generators are all the elements of $\mathcal{H}$ and relations are all the identities hold in $\mathcal{H}$. The *faithful* completions (whose containing a generating copy of $\mathcal{H}$) are of particular importance. To expose a similarity with (I) we associate with a faithful completion X a combinatorial object $\Xi = \Xi(X, \mathcal{H})$ known as the *coset geometry* on which X induces a flag-transitive action. This construction equips some group theoretical notions with topological meaning: the homomorphisms of faithful completions correspond to local isomorphisms of the coset geometries; if X is the universal completion of $\mathcal{H}$, then $\Xi(X, \mathcal{H})$ is simply connected and vice versa. The ideal outcome is when the group H we are after is the universal completion of its subamalgam $\mathcal{H}$. In the classical situation, this is always the case whenever H is taken to be the universal central cover of a finite simple group of Lie type of rank at least 3 and $\mathcal{H}$ is the amalgam of parabolic subgroups containing a given Borel subgroup.

By the classification of flag-transitive Petersen and tilde geometries accomplished in [Iv99] and [ISh02], the Monster is the universal completion of an amalgam formed by a triple of subgroups

$$G_1 \sim 2_+^{1+24}.Co_1,$$
$$G_2 \sim 2^{2+11+22}.(M_{24} \times S_3),$$
$$G_3 \sim 2^{3+6+12+18}.(3 \cdot S_6 \times L_3(2)),$$

where $[G_2 : G_1 \cap G_2] = 3,\ [G_3 : G_1 \cap G_3] = [G_3 : G_2 \cap G_3] = 7$. In fact, explicitly or implicitly, this amalgam has played an essential role in proofs of all principal results about the Monster, including discovery, construction, uniqueness, subgroup structure, Y-theory, moonshine theory.

The purpose of this book is to build up the foundation of the theory of the Monster group adopting the amalgam formed by G_1, G_2, and G_3 as the first principle. The strategy is similar to that followed for the fourth Janko group J_4 in [Iv04] and it amounts to accomplishing the following principal steps:

(A) ‘cut out’ the subset $G_1 \cup G_2 \cup G_3$ from the Monster group and axiomatize the partially defined multiplication to obtain an abstract *Monster amalgam* $\mathcal{M}$;

(B) deduce from the axioms of $\mathcal{M}$ that it exists and is unique up to isomorphism;

(C) by constructing a faithful (196 883-dimensional) representation of $\mathcal{M}$ establish the existence of a faithful completion;

(D) show that a particular subamalgam in $\mathcal{M}$ possesses a unique faithful completion which is the (non-split) extension $2 \cdot BM$ of the group of order 2 by the Baby Monster sporadic simple group BM (this proves that every faithful completion of $\mathcal{M}$ contains $2 \cdot BM$ as a subgroup);

(E) by enumerating the suborbits in a graph on the cosets of the $2 \cdot BM$-subgroup in a faithful completion of $\mathcal{M}$ (known as the *Monster graph*), show that for any such completion the number of cosets is the same (equal to the index of $2 \cdot BM$ in the Monster group);

(F) defining G to be the universal completion of $\mathcal{M}$ conclude that G is the Monster as we know it, that is a non-abelian simple group, in which G_1 is the centralizer of an involution and that

$$|G| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$

In terms of the Monster group G, the Monster graph can be defined as the graph on the class of $2A$-involutions in which two involutions are adjacent if and only if their product is again a $2A$-involution. The centralizer in G of a $2A$-involution is just the above-mentioned subgroup $2 \cdot BM$. It was known for a long time that the $2A$-involutions in the Monster form a class of 6-transpositions in the sense that the product of any two such involutions has order at most 6. At the same time the $2A$-involutions act on the 196 884-dimensional G-module in a very specific manner, in particular we can establish a G-invariant correspondence of the $2A$-involutions with a family of so-called *axial vectors* so that the action of an involution is described by some simple rules formulated in terms of the axial vector along with the G-invariant inner and algebra products on this module (the latter product goes under the name of *Griess algebra*). The subalgebras in the Griess algebra generated by pairs of axial vectors were calculated by Simon Norton [N96]: there are nine isomorphism types and the dimension is at most eight. By a remarkable result recently proved by Shinya Sakuma in the framework of the Vertex Operator Algebras [Sak07], these nine types as well as the 6-transposition property are implied by certain properties of the axial vectors and the corresponding involutions. In this volume we axiomatize these properties under the names of *Majorana axial vectors* and *Majorana involutions*. The fact that the Monster is generated by Majorana involutions will certainly dominate the future studies.