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978-0-521-88994-0 - The Monster Group and Majorana Involutions

A. A. Ivanov

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CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521889940

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First published 2009

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data

ISBN 978-0-521-88994-0 hardback

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To Love and Nina

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Preface

The *Monster* is the most amazing among the finite simple groups. The best way to approach it is via an amalgam called the *Monster amalgam*.

Traditionally one of the following three strategies are used in order to construct a finite simple group H :

- (I) realize H as the automorphism group of an object Ξ ;
- (II) define H in terms of generators and relations;
- (III) identify H as a subgroup in a ‘familiar’ group F generated by given elements.

The strategy offered by the *amalgam method* is a symbiosis of the above three. Here the starting point is a carefully chosen generating system $\mathcal{H} = \{H_i \mid i \in I\}$ of subgroups in H . This system is being axiomatized under the name of *amalgam* and for a while lives a life of its own independently of H . In a sense this is almost like (III) although there is no ‘global’ group F (familiar or non-familiar) in which the generation takes place. Instead one considers the class of all *completions* of \mathcal{H} which are groups containing a quotient of \mathcal{H} as a generating set. The axioms of \mathcal{H} as an abstract amalgam do not guarantee the existence of a completion which contains an isomorphic copy of \mathcal{H} . This is a familiar feature of (II): given generators and relations it is impossible to say in general whether the defined group is trivial or not. This analogy goes further through the *universal completion* whose generators are all the elements of \mathcal{H} and relations are all the identities hold in \mathcal{H} . The *faithful* completions (whose containing a generating copy of \mathcal{H}) are of particular importance. To expose a similarity with (I) we associate with a faithful completion X a combinatorial object $\Xi = \Xi(X, \mathcal{H})$ known as the *coset geometry* on which X induces a flag-transitive action. This construction equips some group theoretical notions with topological meaning: the homomorphisms of faithful completions correspond to local isomorphisms of the coset geometries; if X is the universal completion

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of \mathcal{H} , then $\Xi(X, \mathcal{H})$ is simply connected and vice versa. The ideal outcome is when the group H we are after is the universal completion of its subamalgam \mathcal{H} . In the classical situation, this is always the case whenever H is taken to be the universal central cover of a finite simple group of Lie type of rank at least 3 and \mathcal{H} is the amalgam of parabolic subgroups containing a given Borel subgroup.

By the classification of flag-transitive Petersen and tilde geometries accomplished in [Iv99] and [ISh02], the Monster is the universal completion of an amalgam formed by a triple of subgroups

$$\begin{aligned} G_1 &\sim 2_+^{1+24}.Co_1, \\ G_2 &\sim 2^{2+11+22}.(M_{24} \times S_3), \\ G_3 &\sim 2^{3+6+12+18}.(3 \cdot S_6 \times L_3(2)), \end{aligned}$$

where $[G_2 : G_1 \cap G_2] = 3$, $[G_3 : G_1 \cap G_3] = [G_3 : G_2 \cap G_3] = 7$. In fact, explicitly or implicitly, this amalgam has played an essential role in proofs of all principal results about the Monster, including discovery, construction, uniqueness, subgroup structure, Y -theory, moonshine theory.

The purpose of this book is to build up the foundation of the theory of the Monster group adopting the amalgam formed by G_1 , G_2 , and G_3 as the first principle. The strategy is similar to that followed for the fourth Janko group J_4 in [Iv04] and it amounts to accomplishing the following principal steps:

- (A) ‘cut out’ the subset $G_1 \cup G_2 \cup G_3$ from the Monster group and axiomatize the partially defined multiplication to obtain an abstract *Monster amalgam* \mathcal{M} ;
- (B) deduce from the axioms of \mathcal{M} that it exists and is unique up to isomorphism;
- (C) by constructing a faithful (196 883-dimensional) representation of \mathcal{M} establish the existence of a faithful completion;
- (D) show that a particular subamalgam in \mathcal{M} possesses a unique faithful completion which is the (non-split) extension $2 \cdot BM$ of the group of order 2 by the Baby Monster sporadic simple group BM (this proves that every faithful completion of \mathcal{M} contains $2 \cdot BM$ as a subgroup);
- (E) by enumerating the suborbits in a graph on the cosets of the $2 \cdot BM$ -subgroup in a faithful completion of \mathcal{M} (known as the *Monster graph*), show that for any such completion the number of cosets is the same (equal to the index of $2 \cdot BM$ in the Monster group);

- (F) defining G to be the universal completion of \mathcal{M} conclude that G is the Monster as we know it, that is a non-abelian simple group, in which G_1 is the centralizer of an involution and that

$$|G| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$

In terms of the Monster group G , the Monster graph can be defined as the graph on the class of $2A$ -involutions in which two involutions are adjacent if and only if their product is again a $2A$ -involution. The centralizer in G of a $2A$ -involution is just the above-mentioned subgroup $2 \cdot BM$. It was known for a long time that the $2A$ -involutions in the Monster form a class of 6-transpositions in the sense that the product of any two such involutions has order at most 6. At the same time the $2A$ -involutions act on the 196 884-dimensional G -module in a very specific manner, in particular we can establish a G -invariant correspondence of the $2A$ -involutions with a family of so-called *axial vectors* so that the action of an involution is described by some simple rules formulated in terms of the axial vector along with the G -invariant inner and algebra products on this module (the latter product goes under the name of *Griess algebra*). The subalgebras in the Griess algebra generated by pairs of axial vectors were calculated by Simon Norton [N96]: there are nine isomorphism types and the dimension is at most eight. By a remarkable result recently proved by Shinya Sakuma in the framework of the Vertex Operator Algebras [Sak07], these nine types as well as the 6-transposition property are implied by certain properties of the axial vectors and the corresponding involutions. In this volume we axiomatize these properties under the names of *Majorana axial vectors* and *Majorana involutions*. The fact that the Monster is generated by Majorana involutions will certainly dominate the future studies.