CHAPTER 1

INTRODUCTION

As developed in [49], stability theory is based on the notion of an *invariant* type, more specifically a *definable type*, and the closely related theory of *independence of substructures*. We will review the definitions in Chapter 2 below; suffice it to recall here that an (absolutely) invariant type gives a recipe yielding, for any substructure A of any model of T, a type $p|A$, in a way that respects elementary maps between substructures; in general one relativizes to a set C of parameters, and considers only A containing C. Stability arose in response to questions in pure model theory, but has also provided effective tools for the analysis of algebraic and geometric structures. The theories of algebraically and differentially closed fields are stable, and the stability-theoretic analysis of types in these theories provides considerable information about algebraic and differential-algebraic varieties. The model companion of the theory of fields with an automorphism is not quite stable, but satisfies the related hypothesis of simplicity; in an adapted form, the theory of independence remains valid and has served well in applications to difference fields and definable sets over them. On the other hand, such tools have played a rather limited role, so far, in o-minimality and its applications to real geometry.

Where do valued fields lie? Classically, local fields are viewed as closely analogous to the real numbers. We take a "geometric" point of view however, in the sense of Weil, and adopt the model completion as the setting for our study. This is Robinson's theory ACVF of algebraically closed valued fields. We will view valued fields as substructures of models of ACVF. Moreover, we admit other substructures involving imaginary elements, notably codes for lattices; these have been classified in [12]. This will be essential not only for increasing the strength of the statements, but even for formulating our basic definitions.

A glance at ACVF reveals immediately a stable part, the residue field k; and an o-minimal part, the value group Γ. Both are stably embedded, and have the induced structure of an algebraically closed field, and an ordered divisible abelian group, respectively. But they amount between them to a small part of the theory. For instance, over the uncountable field $\mathbb{Q}_p$, the residue field has only finitely many definable points, and both k and Γ are countable in the model $\mathbb{Q}_p^a$. As observed by Thomas Scanlon [46], ACVF is not stable over Γ, in the sense of [48].

We seek to show nevertheless that stability-theoretic ideas can play a significant role in the description of valued fields. To this end we undertake two logically independent but mutually motivating endeavors. In Part I we introduce an extension of stability theory. We consider theories that have a stable part, define the notion of a *stably dominated* type, and study its properties. The idea is that a type can be controlled by a very small part, lying in the stable part; by analogy, (but it is more than an analogy), a power series is controlled, with respect to the question of invertibility for instance, by its constant coefficient. Given a large model $\mathcal{U}$ and a set of parameters C from $\mathcal{U}$, we define St_C to be a many-sorted structure whose sorts are the C-definable stably embedded stable subsets of the universe. The basic relations of St_C are those given by C-definable relations of $\mathcal{U}$. Then $\mathrm{St}_C(A)$ (the stable part of A) is the definable closure of A in St_C. We write $A \mathop{\smile\!\!\!\!\!\mid}^{d}_{C} B$ if $\mathrm{St}_C(A) \mathop{\smile\!\!\!\!\!\mid} \mathrm{St}_C(B)$ in the stable structure St_C and $\mathrm{tp}(B/C\,\mathrm{St}_C(A)) \vdash \mathrm{tp}(B/CA)$, and say that $\mathrm{tp}(A/C)$ is *stably dominated* if, for all B, whenever $\mathrm{St}_C(A) \mathop{\smile\!\!\!\!\!\mid} \mathrm{St}_C(B)$, we have $A \mathop{\smile\!\!\!\!\!\mid}^{d}_{C} B$. In this case $\mathrm{tp}(A/\mathrm{acl}(C))$ lifts uniquely to an $\mathrm{Aut}(\mathcal{U}/\mathrm{acl}(C))$-invariant type p. Base-change results (under an extra assumption of existence of invariant extensions of types) show that if p is also $\mathrm{Aut}(\mathcal{U}/\mathrm{acl}(C'))$-invariant then $p|C'$ is stably dominated; hence, under this assumption, stable domination is in fact a property of this invariant type, and not of the particular base set. We formulate a general notion of domination-equivalence of invariant types (2.2). In these terms, an invariant type is stably dominated iff it is domination-equivalent to a type of elements in a stable part St_C.

Essentially the whole forking calculus becomes available for stably dominated types. Properties such as definability, symmetry, transitivity, characterization in terms of dividing, lift easily from St_C to $\mathop{\smile\!\!\!\!\!\mid}^{d}$. Others, notably the descent part of base change, require more work and in fact an additional assumption: that for any algebraically closed substructure $C \subseteq M \models T$, any type p over C extends to an $\mathrm{Aut}(M/C)$-invariant type p' over M.

We isolate a further property of definable types in stable theories. Two functions are said to have the same *germ* relative to an invariant type p if they agree generically on p. In the o-minimal context, an example of this is the germ at ∞ of a function on $\mathbb{R}$. Moving from the function to the germ one is able to abstract away from the artifacts of a particular definition. In stability, this is an essential substitute for a topology. For instance, if f is a function into a sort D, one shows that the germ is internal to D; this need not be the case for a code for the function itself. In many stable applications, the strength of this procedure depends on the ability to reconstruct a representative of the germ from the germ alone. We say that a germ is *strong* if this is the case.

It is easy to see the importance of strong germs for the coding of imaginaries. One wants to code a function; as a first approximation, code the germ of the function; if the code is strong, one has succeeded in coding at least a (generic) piece of the function in question. If it is not, one seems to have nothing at all.

We show that germs of stably dominated invariant types are always strong. The proof depends on a combinatorial lemma saying that finite set functions on pairs, with a certain triviality property on triangles, arise from a function on singletons; in this sense it evokes a kind of primitive 2-cohomology, rather as the fundamental combinatorial lemma behind simplicity has a feel of 2-homology. Curiously, both can be proved using the fundamental lemma of stability.

In [15] is is shown that stable domination works well with definable groups. A group G is called generically metastable if it has a translation invariant stably dominated definable type. In this case there exists a unique translation invariant definable type; and the stable domination can be witnessed by a definable homomorphism $h\colon G \to H$ onto a connected stable definable group. Conversely, given such a homomorphism h, G is generically metastable iff the fiber of h above a generic element of H is a complete type. Equivalently, for any definable subset R of G, the set Y of elements $y \in H$ such that $h^{-1}(y)$ is neither contained in, nor disjoint from R is a small set; no finite union of translates of Y covers H. We show this in Theorem 6.13, again using strong germs.

The general theory is at present developed locally, at the level of a single type. It is necessary to say when we expect it to be meaningful globally. The condition cannot be that every type be stably dominated; this would imply stability. Instead we would like to say that uniformly definable families of stably dominated types capture, in some sense, all types. Consider theories with a distinguished predicate Γ, that we assume to be linearly ordered so as to sharply distinguish it from the stable part. We define a theory to be *metastable over* Γ (Definition 4.11) if every type over an algebraically closed set extends to an invariant type, and, over sufficiently rich base sets, every type falls into a Γ-parameterized family of stably dominated types. We show that this notion is preserved under passage to imaginary sorts.

The proviso of "sufficiently rich base set" is familiar from stability, where the primary domination results are valid only over sufficiently saturated models; a great deal of more technical work is then needed to obtain some of them over arbitrary base. The saturation requirement (over "small" base sets) is effective since types over a model are always based on a small set. In the metastable context, more global conditions incompatible with stability are preferred. This will be discussed for ACVF below.

For some purposes, extensions of the base are harmless and the theory can be used directly. This is so for results asserting the existence of a canonical definable set or relation of some kind, since a posteriori the object in question is defined without extra parameters. This occurred in the classification of maps from Γ in [12]. Another instance is in [15], where under certain finiteness of rank assumptions, it is shown that a metastable Abelian group is an extension of a group interpretable over Γ by a definable direct limit of generically metastable groups.

In Part II we study ACVF. This is a C-minimal theory, in the sense of [35], [13]: there exists a uniformly definable family of equivalence relations, linearly ordered by refinement; their classes are referred to as (ultrametric) *balls*; and any definable set (in 1-space) is a Boolean combination of balls. In strongly minimal and o-minimal contexts, one often argues by induction on dimension, fibering an n-dimensional set over an $n-1$-dimensional set with 1-dimensional fibers, thus reducing many questions to the one-dimensional case over parameters. This can also be done in the C-minimal context. Let us call this procedure "dévissage".

A difficulty arises: many such arguments require canonical parameters, not available in the field sort alone. And certainly all our notions, from algebraic closure to stable embeddedness, must be understood with imaginaries. The imaginary sorts of ACVF were given concrete form in [12]: the spaces S_n of n-dimensional lattices, and certain spaces T_n, fibered over S_n with fibers isomorphic to finite dimensional vector spaces over the residue field. But though concrete, these are not in any sense one-dimensional; attempting to reduce complexity by induction on the number of coordinates only leads to subsets of S_n, which is hardly simpler than $(S_n)^m$.

Luckily, S_n itself admits a sequence of fibrations $S_n = X_N \to X_{N-1} \to \cdots \to X_0$, with X_0 a point and such that the fibers of $X_{i+1} \to X_i$ are o- or C-minimal. This uses the transitive action of the solvable group of upper triangular matrices on S_n; see the paragraph following Proposition 7.14. There is a similar statement for T_n (where strongly minimal fibers also occur.) It follows that any definable set of imaginaries admits a sequence of fibrations with successive fibers that are strongly, o- or C-minimal ("unary sets"), or finite. Dévissage arguments are thus possible.

One result obtained this way is the existence of invariant extensions. A type over a base set C can only have an invariant extension if it is *stationary*, i.e., implies a complete type over $\mathrm{acl}(C)$. We show that in ACVF, every stationary type over C has an $\mathrm{Aut}(\mathcal{U}/C)$-invariant extension. For C-minimal sets (including strongly minimal and o-minimal ones), there is a standard choice of invariant extension: the extension avoiding balls of radius smaller than necessary.

But this does not suffice to set up an induction, since for finite sets there is no invariant extension at all. Thus a minimal step of induction consists of *finite covers of C-minimal sets*, i.e., with sets Y admitting a finite-to-one map

$\pi\colon Y \to X$, with X unary. This is quite typical of ACVF, and resembles algebraic geometry, where dévissage can reduce as far as *curves* but not to a single variable. In the o-minimal case, by contrast, one can do induction on ambient dimension, or the number of coordinates of a tuple; this explains much of the more "elementary" feel of basic o-minimality vs. strong minimality.

The additional ingredient needed to obtain invariant extensions of types is the *stationarity lemma* from [12], implying that if π admits a section over a larger base, then it admits a section over $\mathrm{acl}(C)$. See Lemma 8.10. For the theory ACF over a perfect field, stationarity corresponds to the notion of a regular extension, and the stationarity lemma to the existence of a geometric notion of irreducibility of varieties. It is instructive to recall the proof for ACVF. Given a finite cover $\pi\colon Y \to X$ as above, a section s of Y will have a strong germ with respect to the canonical invariant extension of any type of X. Generic types of closed balls are stably dominated; for these, by the results of Part I, all functions have strong germs. Other types are viewed as limits of definable maps from Γ into the space of generics of closed balls. For instance if $\tilde{b}$ is an open ball, consider the family of closed sub-balls b of $\tilde{b}$; these can be indexed by their radius $\gamma \in \Gamma$ the moment one fixes a point in $\tilde{b}$; by the stably dominated case, one has a section of π over each b. The classification of definable maps from Γ (actually from finite covers of Γ) is then used to glue them into a single section, over the original base. This could be done abstractly for C-minimal theories whose associated (local) linear orderings satisfy $\mathrm{dcl}(\Gamma) = \mathrm{acl}(\Gamma)$. The proof of elimination of imaginaries itself has a similar structure; see a sketch at the end of Chapter 15.

Another application of the unary decomposition is the existence of canonical resolutions, or prime models. In the field sorts, ACVF has prime models trivially; the prime model over a nontrivially valued field F is just the algebraic closure F^{alg}. In the geometric sorts the situation becomes more interesting. The algebraic closure does not suffice, but we show that finitely generated structures (or structures finitely generated over models) do admit canonical prime models. A key point is that the prime model over a finitely generated structure A add to A no elements of the residue field or value group. This is important in the theory of motivic integration; see the discussion of resolution in [19]. A further application of canonical resolution is a quantifier-elimination for $\mathbb{C}((t))$ in the $\mathcal{G}$-sorts, relative to the value group Γ. In essence resolution is used to produce functions on imaginary sorts; in fact for any $\mathcal{G}$-sort represented as X/E and any function h on X into the value group or residue field, there exists a function H on X/E such that $H(u) = h(x)$ for some $x \in X/E$.

The construction of prime models combines the decomposition into unary sets with the idea of opacity. An equivalence relation E on X is called *opaque* if any definable subset of X is a union of classes of E, up to a set contained in finitely many classes. This is another manifestation of a recurring theme.

Given $f : X \to Y$ and an ideal I on Y, we say X is dominated by Y via (f, I) if for any subset R of X, for I-almost every $y \in Y$, the fiber $f^{-1}(y)$ is contained in R or is disjoint from it. For stable domination, Y is stable and I is the forking ideal; for stationarity, f has finite fibers, and I is the dual ideal to an invariant type; for opacity, I is the ideal of finite sets. The equivalence relations associated with the analyses of S_n and T_n above are opaque. For an opaque equivalence relation, all elements in a non-algebraic class have the same type (depending only on the class); this gives a way to choose elements in such a non-algebraic class canonically up to isomorphism. Algebraic classes are dealt with in another way.

We now discuss the appropriate notion of a "sufficiently rich" structure. In the stable part, saturation is the right requirement; this will not actually be felt in the present work, since the stable part is $\aleph_1$-categorical and does not really need saturation. For the o-minimal part, a certain completeness condition turns out to be useful; see Chapter 13.2. It allows the description of the semi-group of invariant types up to domination-equivalence, and a characterization of forking in ACVF over very rich bases. For the most part however neither of these play any role; the significant condition is richness over the stable and the o-minimal parts. Here we adopt Kaplansky's maximally complete fields. An algebraically closed valued is maximally complete if it has no proper immmediate extensions. It follows from ([26], [27]) that any model of ACVF embeds in a maximally complete field, uniquely up to isomorphism. Since we use all the geometric sorts, a 'rich base' for us is a model of ACVF whose field part is maximally complete.

Over such a base C, we prove first, using standard results on finite dimensional vector spaces over maximally complete fields, that any field extension F is dominated by its parts in the residue field $k(F)$ and the value group Γ_F. This kind of domination does not admit descent. A stronger statement is that F is dominated by the stable part *over* C together with Γ_F, so that the type of any element of F^n over $C \cup \Gamma_F$ is stably dominated. After an algebraic interpretation of this statement, it is deduced from the previous one by a perturbation argument. Both these results are then extended to imaginary elements.

We interpret the last result as follows: an arbitrary type lies in a family of stably dominated types, definably indexed by Γ. Note that k and Γ play asymmetric roles here. Indeed, at first approximation, we develop what can be thought of as the model theory of k^Γ, rather than $k \times \Gamma$. However k^Γ is presented by a Γ-indexed system of *opaque* equivalence relations, each hiding the structure on the finer ones until a specific class is chosen. This kind of phenomenon, with hidden forms of k^n given by finitely many nested equivalence relations, is familiar from stability theory; the presence of a definable directed system of levels is new here.

Even for fields, the stable domination in the stronger statement cannot be understood without imaginaries. Consider a field extension F of C; for simplicity suppose the value group Γ_F of F is generated over Γ_C by one element γ. There is then a canonical vector space V_γ over the residue field. If γ is viewed as a code for a closed ball $E_\gamma = \{x : v(x) \geq \gamma\}$, the elements of V_γ can be taken to be codes for the maximal open sub-balls of E_γ. The vector space V_γ lies in the stable part of the theory, over $C(\gamma)$. We show that F is dominated over $C(\gamma)$ by elements of $k(F) \cup V_\gamma(F)$. Note that $k(F)$ may well be empty.

Over arbitrary bases, invariant types orthogonal to the value group are shown to be dominated by their stable part; this follows from existence of invariant extensions, and descent.

At this point, we have the metastability of ACVF. We now seek to relate this still somewhat abstract picture more directly with the geometry of valued fields. We characterize the stably dominated types as those invariant types that are orthogonal to the value group (Chapter 8.) In Chapter 14, we describe geometrically the connection between a stably dominated type P and the associated invariant type p, when P is contained in an algebraic variety V. In the case of ACF, the invariant extension is obtained by avoiding all proper subvarieties. In ACVF, the demand is not only to avoid but to stay as far away as possible from any given subvariety. See Theorem 14.12. In ACF the same prescription yields the unique invariant type of any definable set; it is not necessary to pass through types. In ACVF the picture for general definable sets is more complicated. But for a definable subgroup G of $\mathrm{GL}_n(K)$, or for a definable affine homogeneous space, we show that a translation invariant stably dominated type is unique if it exists, and that in this case it is again the type of elements of maximal distance from any proper subvariety of the Zariski closure of G.

In chapter 15 the ideas are similar, but the focus is on canonical bases. Any definable type, in general, has a smallest substructure over which it is defined. In ACF, this is essentially the field of definition of the associated prime ideal. We obtain a similar geometric description for stably dominated types; the ideal of regular functions vanishing on the type is replaced by the R-module of functions taking small values on it.

While presented here for *stably dominated types*, where the theory flows smoothly from the main ideas, within the text we try to work with weaker hypotheses on the types when possible. Over sufficiently rich base structures, all our results can be read off from the main domination results discussed above. But over smaller bases this is not always the case, leading us to think that perhaps a general principle remains to be discovered. An example is the theorem of Chapter 10, that an indiscernible sequence whose canonical base (in an appropriate sense) is orthogonal to Γ, is in fact an indiscernible set, and indeed a Morley sequence for a stably dominated type. Others are phrased in the language of *independence of substructures*.

Classical stability theory yields a notion of independence of two substructures A, B over their intersection C, defined in many equivalent ways. One is directly connected to invariant types: if A is generated by elements a, then A, B are independent over C iff $\mathrm{tp}(a/B)$ has a C-invariant extension to any model. Intuitively 'A is independent from B over C' should say that 'B provides as little as possible extra information about A, beyond what C provides'. In other words, the locus of A over B is as large as possible inside the locus of A over C. A number of the above ideas lead to notions of independence for substructures of models of ACVF, i.e., for valued fields.

The simplest notion, *sequential independence* (Chapter 8), depends on the choice of an ordered tuple a of generators of A over C. Let p be the invariant extension of $\mathrm{tp}(a/C)$ constructed above by dévissage. We say that A is sequentially independent from B over C, $A \mathop{\smile\!\!\!\!\!\!|}^{g}_{C} B$, if $\mathrm{tp}(a/B) \subset p$. In general, the notion depends on the order of the tuple, and is not symmetric.

A point in an irreducible variety is generic if it does not lie in any smaller dimensional variety over the same parameters, and in an algebraically closed field, A is independent from B over C if every tuple from A which is generic in a variety defined over C remains generic in the same variety with the additional parameters from $C \cup B$. Since varieties are defined by polynomial equations, this is equivalent to saying that for every $a \in A$, the ideal of polynomials which vanish on the $\mathrm{tp}(a/C)$ is the same as those which vanish on $\mathrm{tp}(a/C \cup B)$. We extend both of these points of view to an algebraically closed valued field.

In this setting, the definable sets depend on the valuation as well, so we consider the set of polynomials which satisfy a valuation inequality on $\mathrm{tp}(a/C)$. This is no longer an ideal, but naturally gives a collection of modules over the valuation ring, which we call $J(\mathrm{tp}(a/C))$. We define A to be J-independent from B over C if $J(\mathrm{tp}(a/C)) = J(\mathrm{tp}(a/C \cup B))$ for all tuples a from A (Chapter 15). This definition does not depend on the order of the tuple a.

Our final notion of independence is defined here only for variables of the field sort. We define $\mathrm{tp}^+(A/C)$ to be the positive quantifier-free type of A over C, and say that A and B are *modulus independent* over C if $\mathrm{tp}^+(AB/C)$ is determined by the full types of A and B separately (Chapter 14). In a pure algebraically closed field, the positive type corresponds precisely to the ideal of polynomials which vanish on the type, so modulus independence is in that setting another way of stating non-forking. In an algebraically closed valued field, we use modulus independence as a step from sequential independence to J-independence. In this setting, the quantifier-free positive formulas refer to the maximum norm that a polynomial can take on a type.

All these notions agree on stably dominated types, and have the good properties of independence for stable theories (see Theorem 15.9). Under more general conditions, they diverge, and various properties can fail; for instance, for non-stably dominated types, J-independence need not be symmetric in A and B, nor need $\mathrm{tp}(A/C)$ have a J-independent extension over $C \cup B$. We give numerous examples showing this. We do show however that if A and B are fields, and $C \cap K \leq A$ with $\Gamma(C) = \Gamma(A)$, then sequential independence over C implies both modulus independence and J-independence.

In the final chapter we briefly illustrate the idea mentioned in the preface, that the methods of this monograph should be useful for valued fields beyond ACVF. Theorem 16.7 asserts that Scanlon's model completion of valued differential fields is metastable. This gives for the first time a language to pose structural questions about this rich theory, and we hope it will be fruitful. We also show the metastability of the theory of $\mathbb{C}((t))$, and related theories. Here we prove nothing anew, reducing all questions to properties of ACVF. The property of metastability itself can only hold for a limited number of theories of valued fields, but the method of reduction to the algebraic closure is much more general.

PART 1

STABLE DOMINATION