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### **An Introduction to Lie Groups and Lie Algebras**

With roots in the nineteenth century, Lie theory has since found many and varied applications in mathematics and mathematical physics, to the point where it is now regarded as a classical branch of mathematics in its own right. This graduate text focuses on the study of semisimple Lie algebras, developing the necessary theory along the way.

The material covered ranges from basic definitions of Lie groups, to the theory of root systems, and classification of finite-dimensional representations of semisimple Lie algebras. Written in an informal style, this is a contemporary introduction to the subject which emphasizes the main concepts of the proofs and outlines the necessary technical details, allowing the material to be conveyed concisely.

Based on a lecture course given by the author at the State University of New York at Stony Brook, the book includes numerous exercises and worked examples and is ideal for graduate courses on Lie groups and Lie algebras.

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# An Introduction to Lie Groups and Lie Algebras

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*Dedicated to my teachers*

## Contents

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	<i>Preface</i>	<i>page xi</i>
<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Lie groups: basic definitions</b>	<b>4</b>
	2.1. Reminders from differential geometry	4
	2.2. Lie groups, subgroups, and cosets	5
	2.3. Lie subgroups and homomorphism theorem	10
	2.4. Action of Lie groups on manifolds and representations	10
	2.5. Orbits and homogeneous spaces	12
	2.6. Left, right, and adjoint action	14
	2.7. Classical groups	16
	2.8. Exercises	21
<b>3</b>	<b>Lie groups and Lie algebras</b>	<b>25</b>
	3.1. Exponential map	25
	3.2. The commutator	28
	3.3. Jacobi identity and the definition of a Lie algebra	30
	3.4. Subalgebras, ideals, and center	32
	3.5. Lie algebra of vector fields	33
	3.6. Stabilizers and the center	36
	3.7. Campbell–Hausdorff formula	38
	3.8. Fundamental theorems of Lie theory	40
	3.9. Complex and real forms	44
	3.10. Example: $\mathfrak{so}(3, \mathbb{R})$ , $\mathfrak{su}(2)$ , and $\mathfrak{sl}(2, \mathbb{C})$	46
	3.11. Exercises	48

<b>4</b>	<b>Representations of Lie groups and Lie algebras</b>	<b>52</b>
4.1.	Basic definitions	52
4.2.	Operations on representations	54
4.3.	Irreducible representations	57
4.4.	Intertwining operators and Schur's lemma	59
4.5.	Complete reducibility of unitary representations: representations of finite groups	61
4.6.	Haar measure on compact Lie groups	62
4.7.	Orthogonality of characters and Peter–Weyl theorem	65
4.8.	Representations of $\mathfrak{sl}(2, \mathbb{C})$	70
4.9.	Spherical Laplace operator and the hydrogen atom	75
4.10.	Exercises	80
<b>5</b>	<b>Structure theory of Lie algebras</b>	<b>84</b>
5.1.	Universal enveloping algebra	84
5.2.	Poincaré–Birkhoff–Witt theorem	87
5.3.	Ideals and commutant	90
5.4.	Solvable and nilpotent Lie algebras	91
5.5.	Lie's and Engel's theorems	94
5.6.	The radical. Semisimple and reductive algebras	96
5.7.	Invariant bilinear forms and semisimplicity of classical Lie algebras	99
5.8.	Killing form and Cartan's criterion	101
5.9.	Jordan decomposition	104
5.10.	Exercises	106
<b>6</b>	<b>Complex semisimple Lie algebras</b>	<b>108</b>
6.1.	Properties of semisimple Lie algebras	108
6.2.	Relation with compact groups	110
6.3.	Complete reducibility of representations	112
6.4.	Semisimple elements and toral subalgebras	116
6.5.	Cartan subalgebra	119
6.6.	Root decomposition and root systems	120
6.7.	Regular elements and conjugacy of Cartan subalgebras	126
6.8.	Exercises	130
<b>7</b>	<b>Root systems</b>	<b>132</b>
7.1.	Abstract root systems	132
7.2.	Automorphisms and the Weyl group	134
7.3.	Pairs of roots and rank two root systems	135

## Contents

ix

7.4.	Positive roots and simple roots	137
7.5.	Weight and root lattices	140
7.6.	Weyl chambers	142
7.7.	Simple reflections	146
7.8.	Dynkin diagrams and classification of root systems	149
7.9.	Serre relations and classification of semisimple Lie algebras	154
7.10.	Proof of the classification theorem in simply-laced case	157
7.11.	Exercises	160
<b>8</b>	<b>Representations of semisimple Lie algebras</b>	<b>163</b>
8.1.	Weight decomposition and characters	163
8.2.	Highest weight representations and Verma modules	167
8.3.	Classification of irreducible finite-dimensional representations	171
8.4.	Bernstein–Gelfand–Gelfand resolution	174
8.5.	Weyl character formula	177
8.6.	Multiplicities	182
8.7.	Representations of $\mathfrak{sl}(n, \mathbb{C})$	183
8.8.	Harish–Chandra isomorphism	187
8.9.	Proof of Theorem 8.25	192
8.10.	Exercises	194
	<b>Overview of the literature</b>	<b>197</b>
	Basic textbooks	197
	Monographs	198
	Further reading	198
	<b>Appendix A Root systems and simple Lie algebras</b>	<b>202</b>
A.1.	$A_n = \mathfrak{sl}(n + 1, \mathbb{C}), n \geq 1$	202
A.2.	$B_n = \mathfrak{so}(2n + 1, \mathbb{C}), n \geq 1$	204
A.3.	$C_n = \mathfrak{sp}(n, \mathbb{C}), n \geq 1$	206
A.4.	$D_n = \mathfrak{so}(2n, \mathbb{C}), n \geq 2$	207
	<b>Appendix B Sample syllabus</b>	<b>210</b>
	<b>List of notation</b>	<b>213</b>
	<i>Bibliography</i>	216
	<i>Index</i>	220



## Preface

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This book is an introduction to the theory of Lie groups and Lie algebras, with emphasis on the theory of semisimple Lie algebras. It can serve as a basis for a two-semester graduate course or – omitting some material – as a basis for a rather intensive one-semester course. The book includes a large number of exercises.

The material covered in the book ranges from basic definitions of Lie groups to the theory of root systems and highest weight representations of semisimple Lie algebras; however, to keep book size small, the structure theory of semisimple and compact Lie groups is not covered.

Exposition follows the style of famous Serre's textbook on Lie algebras [47]: we tried to make the book more readable by stressing ideas of the proofs rather than technical details. In many cases, details of the proofs are given in exercises (always providing sufficient hints so that good students should have no difficulty completing the proof). In some cases, technical proofs are omitted altogether; for example, we do not give proofs of Engel's or Poincaré–Birkhoff–Witt theorems, instead providing an outline of the proof. Of course, in such cases we give references to books containing full proofs.

It is assumed that the reader is familiar with basics of topology and differential geometry (manifolds, vector fields, differential forms, fundamental groups, covering spaces) and basic algebra (rings, modules). Some parts of the book require knowledge of basic homological algebra (short and long exact sequences, Ext spaces).

Errata for this book are available on the book web page at <http://www.math.sunysb.edu/~kirillov/liegroups/>.