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Introduction

In any algebra textbook, the study of group theory is usually mostly concerned with the theory of finite, or at least finitely generated, groups. This is understandable: such groups are much easier to describe. However, most groups which appear as groups of symmetries of various geometric objects are not finite: for example, the group $SO(3, \mathbb{R})$ of all rotations of three-dimensional space is not finite and is not even finitely generated. Thus, much of material learned in basic algebra course does not apply here; for example, it is not clear whether, say, the set of all morphisms between such groups can be explicitly described.

The theory of Lie groups answers these questions by replacing the notion of a finitely generated group by that of a Lie group – a group which at the same time is a finite-dimensional manifold. It turns out that in many ways such groups can be described and studied as easily as finitely generated groups – or even easier. The key role is played by the notion of a Lie algebra, the tangent space to G at identity. It turns out that the group operation on G defines a certain bilinear skew-symmetric operation on $\mathfrak{g} = T_1 G$; axiomatizing the properties of this operation gives a definition of a Lie algebra.

The fundamental result of the theory of Lie groups is that many properties of Lie groups are completely determined by the properties of corresponding Lie algebras. For example, the set of morphisms between two (connected and simply connected) Lie groups is the same as the set of morphisms between the corresponding Lie algebras; thus, describing them is essentially reduced to a linear algebra problem.

Similarly, Lie algebras also provide a key to the study of the structure of Lie groups and their representations. In particular, this allows one to get a complete classification of a large class of Lie groups (semisimple and more generally, reductive Lie groups; this includes all compact Lie groups and all classical Lie groups such as $SO(n, \mathbb{R})$) in terms of relatively simple geometric objects, so-called root systems. This result is considered by many mathematicians (including the author of this book) to be one of the most beautiful achievements in all of mathematics. We will cover it in Chapter 7.

To conclude this introduction, we will give a simple example which shows how Lie groups naturally appear as groups of symmetries of various objects – and how one can use the theory of Lie groups and Lie algebras to make use of these symmetries.

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. Define the Laplace operator $\Delta_{\mathrm{sph}} : C^\infty(S^2) \to C^\infty(S^2)$ by $\Delta_{\mathrm{sph}} f = (\Delta \tilde{f})|_{S^2}$, where $\tilde{f}$ is the result of extending f to $\mathbb{R}^3 - \{0\}$ (constant along each ray), and Δ is the usual Laplace operator in $\mathbb{R}^3$. It is easy to see that Δ_{sph} is a second-order differential operator on the sphere; one can write explicit formulas for it in the spherical coordinates, but they are not particularly nice.

For many applications, it is important to know the eigenvalues and eigenfunctions of Δ_{sph}. In particular, this problem arises in quantum mechanics: the eigenvalues are related to the energy levels of a hydrogen atom in quantum mechanical description. Unfortunately, trying to find the eigenfunctions by brute force gives a second-order differential equation which is very difficult to solve.

However, it is easy to notice that this problem has some symmetry – namely, the group $\mathrm{SO}(3, \mathbb{R})$ acting on the sphere by rotations. How can one use this symmetry?

If we had just one symmetry, given by some rotation $R \colon S^2 \to S^2$, we could consider its action on the space of complex-valued functions $C^\infty(S^2, \mathbb{C})$. If we could diagonalize this operator, this would help us study Δ_{sph}: it is a general result of linear algebra that if A, B are two commuting operators, and A is diagonalizable, then B must preserve eigenspaces for A. Applying this to pair R, Δ_{sph}, we get that Δ_{sph} preserves eigenspaces for R, so we can diagonalize Δ_{sph} independently in each of the eigenspaces.

However, this will not solve the problem: for each individual rotation R, the eigenspaces will still be too large (in fact, infinite-dimensional), so diagonalizing Δ_{sph} in each of them is not very easy either. This is not surprising: after all, we only used one of many symmetries. Can we use all of rotations $R \in \mathrm{SO}(3, \mathbb{R})$ simultaneously?

This, however, presents two problems.

- $\mathrm{SO}(3, \mathbb{R})$ is not a finitely generated group, so apparently we will need to use infinitely (in fact uncountably) many different symmetries and diagonalize each of them.
- $\mathrm{SO}(3, \mathbb{R})$ is not commutative, so different operators from $\mathrm{SO}(3, \mathbb{R})$ can not be diagonalized simultaneously.

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The goal of the theory of Lie groups is to give tools to deal with these (and similar) problems. In short, the answer to the first problem is that $\mathrm{SO}(3, \mathbb{R})$ is in a certain sense finitely generated – namely, it is generated by three generators, "infinitesimal rotations" around x, y, z axes (see details in Example 3.10).

The answer to the second problem is that instead of decomposing the $C^\infty(S^2, \mathbb{C})$ into a direct sum of common eigenspaces for operators $R \in \mathrm{SO}(3, \mathbb{R})$, we need to decompose it into "irreducible representations" of $\mathrm{SO}(3, \mathbb{R})$. In order to do this, we need to develop the theory of representations of $\mathrm{SO}(3, \mathbb{R})$. We will do this and complete the analysis of this example in Section 4.8.