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Excerpt
More information

1
Introduction

A physical measurement requires a collection of devices such as a clock, a theodolite, a counter, a light gun, and so on. The operational control of this instrumentation is exercised by the observer, who decides what to measure, how to perform a measurement, and how to interpret the results. The observer's laboratory covers a finite spatial volume and the measurements last for a finite interval of time so we can define as the *measurement's domain* the space-time region in which a process of measurement takes place. If the background curvature can be neglected, then the measurements will not suffer from curvature effects and will then be termed *local*. On the contrary, if the curvature is strong enough that it cannot be neglected over the measurement's domain, the response of the instruments will depend on the position therein and therefore they require a careful calibration to correct for curvature perturbations. In this case the measurements carrying a signature of the curvature will be termed *non-local*.

1.1 Observers and physical measurements

A laboratory is mathematically modeled by a family of non-intersecting time-like curves having u as tangent vector field and denoted by $\mathcal{C}_u$; this family is also termed the *congruence*. Each curve of the congruence represents the history of a point in the laboratory. We choose the parameter τ on the curves of $\mathcal{C}_u$ so as to make the tangent vector field u unitary; this choice is always possible for non-null curves. Let Σ be a space-like three-dimensional section of C_u spanned by the curves which cross a selected curve γ_* of the congruence orthogonally. The concepts of unitarity and orthogonality are relative to the assumed background metric. The curve γ_* will be termed the *fiducial* curve of the congruence and referred to as the *world line* of the observer. Let the point of intersection of Σ with γ_* be $\gamma_*(\tau)$; as τ varies continuously over γ_*, the section Σ spans a four-dimensional volume which represents the space-time history of the observer's laboratory. Whenever we limit the extension of Σ to a range much smaller than the average radius of its induced curvature, we can identify $\mathcal{C}_u$ with the single curve γ_* and Σ with the point $\gamma_*(\tau)$. Any time-like curve γ with tangent vector u can then be identified as the world line of an *observer*, which will be referred to as "the observer u." If the parameter τ on γ is such as to make the tangent vector unitary, then its physical meaning is that of the *proper time* of the observer u, i.e. the time read on his clock in units of the speed of light in vacuum.

This concept of observer, however, needs to be specialized further, defining a *reference frame* adapted to him. A reference frame is defined by a clock which marks the time as a parameter on γ, as already noted, and by a *spatial* frame made of three space-like directions identified at each point on γ by space-like curves stemming orthogonally from it. While the time direction is uniquely fixed by the vector field u, the spatial directions are defined up to spatial rotations, i.e. transformations which do not change u; obviously there are infinitely many such spatial perspectives.

The result of a physical measurement is mathematically described by a scalar, a quantity which is invariant under general coordinate transformations. A scalar quantity, however, is not necessarily a physical measurement. The latter, in fact, needs to be defined with respect to an observer and in particular to one of the infinitely many spatial frames adapted to him. The aim of the relativistic theory of measurement is to enable one to devise, out of the tensorial representation of a physical system and with respect to a given frame, those scalars which describe specific properties of the system.

The measurements are in general observer-dependent so, as stated, a criterion should also be given for comparing measurements made by different observers. A basic role in this procedure of comparison is played by the Lorentz group of transformations. A measurement which is observer-independent is termed *Lorentz invariant*. Lorentz invariant measurements are of key importance in physics.

1.2 Interpretation of physical measurements

The description of a physical system depends both on the observer and on the chosen frame of reference. In most cases the result of a measurement is affected by contributions from the background curvature and from the peculiarity of the reference frame. As long as it is not possible to discriminate among them, a measurement remains plagued by an intrinsic ambiguity. We shall present a few examples where this situation arises and discuss possible ways out. The most important among the observer-dependent measurements is that of time intervals. Basic to Einstein's theory of relativity is the relativity of time. Hence we shall illustrate this concept first, dealing with inertial frames for the sake of clarity.

1.3 Clock synchronization and relativity of time

The theory of special relativity, formally issued in 1905 (Einstein, 1905), presupposes that inertial observers are fully equivalent in describing physical laws. This requirement, known as the *principle of relativity*, implies that one has to abandon the concepts of absolute space and absolute time. This step is essential in order to envisage a model of reality which is consistent with observations and in particular with the behavior of light. As is well known, the speed of light c, whose value in vacuum is $2.997\,924\,58 \times 10^5$ km s^{-1}, is independent of the observer who measures it, and therefore is an absolute quantity.

Since time plays the role of a coordinate with the same prerogatives as the spatial ones, one needs a criterion for assigning a value of that coordinate, let us say t, to each space-time point. The criterion of time labeling, also termed *clock synchronization*, should be the same in all frames if we want the principle of relativity to make sense, and this is assured by the universality of the velocity of light. In fact, one uses a light ray stemming from a fiducial point with spatial coordinates x_0, for example, and time coordinate equal to zero, then assigns to each point of spatial coordinates $x_0 + \Delta x$ crossed by the light ray the time $t = \Delta x/c$. In this way, assuming the connectivity of space-time, we can label each of its points with a value of t. Clearly one must be able to fix for each of them the spatial separation Δx from the given fiducial point, but that is a non-trivial procedure which will be discussed later in the book.

The relativity of time is usually stated by saying that if an observer u compares the time t read on his own clock with that read on the clock of an observer u' moving uniformly with respect to u and instantaneously coincident with it, then u finds that t' differs from t by some factor K, as $t' = Kt$.[1] On the other hand, if the comparison is made by the observer u', because of the equivalence of the inertial observers he will find that the time t marked by the clock of u differs from the time t' read on his own clock when they instantaneously coincide, by the *same* factor, as $t = Kt'$. The factor K, which we denote as the *relativity factor*, is at this stage unknown except for the obvious facts that it should be positive, it should depend only on the magnitude of the relative velocity for consistency with the principle of relativity, and finally that it should reduce to one when the relative velocity is equal to zero. Our aim is to find the factor K and explain why it differs in general from one. A similar analysis can be found in Bondi's K-calculus (Bondi, 1980; see also de Felice, 2006). In what follows we shall not require knowledge of the Lorentz transformations nor of any concept of relativity.

Let us consider an inertial frame S with coordinates (x, y, z) and time t. The time axes in S form a congruence of curves each representing the history of a static observer at the corresponding spatial point. Denote by u the fiducial observer of this family, located at the spatial origin of S. At each point of S there exists a clock which marks the time t of that particular event and which would be read by the static observer spatially fixed at that point. All static observers

[1] The choice of a linear relation is justified a posteriori since it leads to the correct theory of relativity.

in S are equivalent to each other since we require that their time runs with the same rate. A clock which is attached to each point of S will be termed an S-clock.

Let S' be another inertial frame with spatial coordinates (x', y', z') and time t'. We require that S' moves uniformly along the x-axis of S with velocity ν. The x-axis of S will be considered spatially coincident with the x'-axis of S', with the further requirement that the origins of x and x' coincide at the time $t = t' = 0$. In this case the relative motion is that of a recession. In the frame S' the totality of time axes forms a congruence of curves each representing the history of a static observer. At each point of S' there is a clock, termed an S'-clock, which marks the time of that particular event and is read by the static observer fixed at the corresponding spatial position. The S'-clocks mark the time t' with the same rate; hence the static observers in S' are equivalent to each other. Finally we denote by u' the fiducial observer of the above congruence of time axes, fixed at the spatial origin of S'.

Let the systems S and S' be represented by the 2-planes (ct, x) and (ct', x') respectively;[2] we then assume that from the spatial origin of S and at time t_u, a light signal is emitted along the x-axis and towards the observer u'. The light signal reaches the observer u' at the time marked by the local S-clock, given by

$$t_{u'} = \frac{t_u}{1 - \nu/c}. \tag{1.1}$$

At this event, the observer u' can read two clocks which are momentarily coincident, namely the S-clock which marks the time $t_{u'}$ as in (1.1) and his own clock which marks a time $t'_{u'}$. In general the time beating on a given clock is driven by a sequence of events; in our case the time read on the clock of the observer u, at the spatial origin of S, follows the emission of the light signals. If these are emitted with continuity[3] then the time marked by the clock of the observer u will be a continuous function on S which we still denote by t_u. The time read on the S-clocks which are *crossed* by the observer u' along his path marks the instants of recording by u' of the light signals emitted by u. The events of reception by u', however, do not belong to the history of one observer only, but each of them, having a different spatial position in S, belongs to the history of the static observer located at the corresponding spatial point.

Let us now consider the same process as seen in the frame S'. The space-time of S' is carpeted by S'-clocks each marking the time t' read by the static observers fixed at each spatial point of S'. The observer u', at the spatial origin of S', receives at time $t'_{u'}$ the light signal emitted by the observer u who is seen receding along the negative direction of the x'-axis. The emission of the light signals by

[2] This is possible without loss of generality because of the homogeneity and isotropy of space.
[3] By continuity here we mean that the time interval between any two events of emission (or of reception) goes to zero.

the observer u occurs at times t'_u read on the S'-clocks *crossed* by u along his path and given by

$$t'_u = \frac{t'_{u'}}{1 + \nu/c}. \tag{1.2}$$

The time on the clock of the observer u', denoted by $t'_{u'}$, runs continuously with the recording of the light signals emitted by u. Meanwhile the observer u can read two clocks, momentarily coincident, namely his own clock which marks a time t_u and the S'-clock which is crossed by u during his motion which marks a time t'_u. Also in this case we have to remember that t'_u is *not* the time read on the clock of one single observer but is the time read at each instant on an S'-clock belonging to the static observer fixed at the corresponding spatial position in S'.

To summarize, the time read on the S-clocks set along the path of u' in S is $t_{u'}$ while the time marked by the clock carried by u' is $t'_{u'}$. Analogously the time read on the S'-clocks set along the path of u in S' is t'_u while the time read by u on his own clock is given by t_u. Our aim is to find the relation between $t'_{u'}$ and $t_{u'}$ in the frame S and that between t_u and t'_u in the frame S'. In both cases we are comparing times read on clocks which are in relative motion but instantaneously coincident.

The observers u and u' located at the spatial origins of S and S' respectively cannot *read* each other's clocks because they will be far apart after the initial time $t = t' = 0$ when they are assumed to coincide. In order to find the relation between t_u and $t'_{u'}$ one has to go through the intermediate steps where

(i) the observer u at the spatial origin of S correlates the time t_u, read on his own clock at the emission of the light signals, to the time $t_{u'}$, marked by the S-clocks when they are reached by the light signals and simultaneously crossed by the observer u' along his path in S;
(ii) the observer u' at the spatial origin of S' correlates the time $t'_{u'}$, read on his own clock, to the time t'_u marked by the S'-clocks when a light signal was emitted and simultaneously crossed by u along his path in S'.

The two points of view are not symmetric; in fact, the light signals are emitted by u and received by u' in both cases. These *intermediate steps* allow us to establish the relativity of time.

The principle of relativity ensures the complete equivalence of the inertial observers in the sense that they will always draw the same conclusions from an equal set of observations. In our case, comparing the points of view of the two observers, we deduce that the ratio between the time $t_{u'}$ that u' reads on each S-clock when he crosses it, and the time $t'_{u'}$ that he reads on his own clock at the same instant, is the same as the ratio between the time t'_u that u reads on each S'-clock which he crosses during his motion in S', and the time t_u that he reads on his own clock at the same instant, namely:

$$\frac{t_{u'}}{t'_{u'}} = \frac{t'_u}{t_u}. \tag{1.3}$$

Taking into account (1.1), relation (1.3) becomes

$$t'_{u'} = t_{u'}\frac{t_u}{t'_u} = \frac{t^2_{u'}}{t'_u}\left(1 - \frac{\nu}{c}\right). \tag{1.4}$$

Then, from (1.2),

$$t'_{u'} = \frac{t^2_{u'}}{t'_{u'}}\left(1 - \frac{\nu^2}{c^2}\right). \tag{1.5}$$

Along the path of u' in S, we have

$$t'_{u'} = \sqrt{1 - \frac{\nu^2}{c^2}}\, t_{u'}. \tag{1.6}$$

Similarly, from (1.3) and (1.2) we have

$$t_u = t'_{u'}\frac{t'_u}{t_{u'}} = \frac{t'^2_u}{t_{u'}}\left(1 + \frac{\nu}{c}\right). \tag{1.7}$$

Hence, from (1.1),

$$t_u = \frac{t'^2_u}{t_u}\left(1 - \frac{\nu^2}{c^2}\right). \tag{1.8}$$

Along the path of u in S' we finally have

$$t_u = \sqrt{1 - \frac{\nu^2}{c^2}}\, t'_u. \tag{1.9}$$

Thus the factor K turns out to be equal to $\sqrt{1 - (\nu/c)^2}$.

The above considerations have been made under the assumption that the observers u and u' are receding from each other. However the above result should still hold if the observers u and u' are approaching instead. We shall prove that this is actually the case.

Indeed the time rates of their clocks depend on the *sense* of the relative motion. In fact, if the two observers move away from each other the light signals emitted by one of them will be seen by the other with a delay, hence at a slower rate, because each signal has to cover a longer path than the previous one. If the observers instead approach each other then the signal emitted by one will be seen by the other with an anticipation due to the relative approaching motion, and so at a faster rate. This is what actually occurs to the time rates of the clocks carried by the observers u and u', namely t_u and $t'_{u'}$. In fact, from (1.6) and (1.1) we deduce, from the point of view of the observer u, that

$$t'_{u'} = \sqrt{\frac{1+\nu/c}{1-\nu/c}}\, t_u. \tag{1.10}$$

Hence, if $\nu > 0$ (u' recedes from u) then $t'_{u'} > t_u$, i.e. u judges the clock of u' to be ticking at a slower rate with respect to his own; if $\nu < 0$ (u' approaching u) then $t'_{u'} < t_u$, that is u now judges the clock of u' to be ticking at a faster rate with respect to his own. Despite this, the difference marked by the clocks of the two frames when they are instantaneously coincident must be independent of the sense of the relative motion. This will be shown in what follows.

Let us consider two frames S and S' approaching each other with velocity ν along the respective coordinate axes x and x'. Let u be the observer at rest at the spatial origin of S and u' the one at rest at the spatial origin of S'. From the point of view of S, the observer u' approaches u along a straight line of equation:

$$x = -\nu t + x_0 \tag{1.11}$$

where x_0 is the spatial position of u' at the initial time $t = 0$. The observer u emits light signals along the x-axis at times t_u. These signals move towards the observer u' and meet him at the events of observation at times $t_{u'}$ read by u' on the S-clocks that he crosses along his path. The equation of motion of the light signals will be in general

$$x = c(t - t_u) \tag{1.12}$$

and so the instant of observation by u' is given by the intersection of the line (1.12), which describes the motion of the light ray, and the line (1.11) which describes that of the observer u', namely

$$c\,(t_{u'} - t_u) = -\nu t_{u'} + x_0 \tag{1.13}$$

which leads to

$$t_{u'} = \frac{t_u + x_0/c}{1+\nu/c}. \tag{1.14}$$

Let us stress what was said before: while times t_u are read on the clock of u at rest in the spatial origin of S, the instants $t_{u'}$ are marked by the S-clocks spatially coincident with the position of the observer u' when he detects the light signals.

Let us now illustrate how the same process is seen in the frame S'. In this case the observer u approaches u' along the axis x' with relative velocity ν and therefore along a straight line of equation

$$x' = \nu t' - x'_0. \tag{1.15}$$

The position of u at the initial time $t' = 0$ is given by some value of the coordinate x' which we set equal to $-x'_0$, with x'_0 positive. This value is related to x_0, which appears in (1.11), by an explicit relation that we here ignore.[4] The observer u sends light signals at times t'_u. These reach the observer u' set in the spatial origin of S' at the instants $t'_{u'}$ read on his own clock. The motion of these signals is described by a straight line whose equation is given in general by

$$x' = c\,(t' - t'_{u'}). \tag{1.16}$$

The time of emission by the observer u is fixed by the intersection of the line (1.16) which describes the motion of the light signal with the line (1.15) which describes the motion of u, namely

$$t'_u = \frac{t'_{u'} - x'_0/c}{1 - \nu/c}. \tag{1.17}$$

Let us recall again here that while $t'_{u'}$ is the time read by u' on his own clock set stably at the spatial origin of S', the time t'_u is marked by the S'-clocks which are instantaneously coincident with the moving observer u. Here we exploit the equivalence between inertial frames regarding the reading of the clocks which lead to (1.3). After some elementary mathematical steps we obtain

$$(t'_{u'})^2 = \left(1 - \frac{\nu^2}{c^2}\right)(t_{u'})^2 - \left(1 - \frac{\nu}{c}\right)\frac{x_0}{c}t_{u'} + \frac{x'_0}{c}t'_{u'}. \tag{1.18}$$

This relation, deduced in the case of approaching observers, does not coincide with the analogous relation (1.6) deduced in the case of observers receding from each other. Although we do not know what the relation between x_0 and x'_0 is, we can prove the symmetry between this case and the previously discussed one.

Let us consider the extension of the light trajectories stemming from u to u' in the frames S and S', until they intersect the world line of static observers located at x_0 and x'_0 respectively. Let us denote these observers as u_0 and u'_0. In the frame S, the light signals intercept the observer u_0 at times, read on the clock of u_0, given by

$$t_{u_0} = t_u + x_0/c. \tag{1.19}$$

Then Eq. (1.14) can also be written as

$$t_{u'} = \frac{t_{u_0}}{1 + \nu/c}. \tag{1.20}$$

The time marked by the clock of u_0 at the arrival of the light signals runs with the same rate as that of the time marked by the clock of u at the emission of the

[4] The quantities x_0 and x'_0 are related by a Lorentz transformation but here we cannot introduce it because the latter presupposes that one knows the relation between the time rates of the spatially coincident clocks of the frames S and S', which instead we want to deduce.

same signals, since u and u_0 have zero relative velocity and therefore are to be considered as the *same* observer located at different spatial positions. Then we can still denote t_{u_0} as t_u and write relation (1.20) as

$$t_{u'} = \frac{t_u}{1 + \nu/c}. \tag{1.21}$$

A similar argument can be repeated in the frame S'. The time read on the clock of u'_0, at the intersections of the light rays with the history of the observer u'_0, is equal to

$$t'_{u'_0} = t'_{u'} - x'_0/c. \tag{1.22}$$

Relation (1.17) can be written as

$$t'_u = \frac{t'_{u'_0}}{1 - \nu/c}. \tag{1.23}$$

The time read on the clock of u'_0 runs at the same rate as that of u' since u'_0 and u' are to be considered as the same observer but located at different spatial positions. From this it follows that (1.23) can also be written as

$$t'_u = \frac{t'_{u'}}{1 - \nu/c}. \tag{1.24}$$

We clearly see that the relative motion of approach of u to u' is equivalent to a relative motion of recession between the observers u' and u_0 in S and between u and u'_0 in S'. The result of this comparison is the same as that shown in the relations (1.6) and (1.9), which are then independent of the sense of the relative motion, as expected. Moreover, this conclusion implies for consistency that, setting in (1.18)

$$t'_{u'} = \sqrt{1 - \frac{\nu^2}{c^2}}\, t_{u'}, \tag{1.25}$$

it follows that

$$-\left(1 - \frac{\nu}{c}\right)\frac{x_0}{c} t_{u'} + \frac{x'_0}{c} t'_{u'} = 0. \tag{1.26}$$

From this and (1.25) we further deduce that

$$x'_0 = \frac{x_0}{\sqrt{1 - \nu^2/c^2}}(1 - \nu/c). \tag{1.27}$$

One should notice here that (1.26) must be solved with respect to x'_0 and *not* with respect x_0 because the corresponding relation (1.25) between times, which implies (1.26), is relative to the situation where the observer u is the one who observes the moving frame S'; hence we have to express all quantities of S' in terms of the coordinates of S. Equations (1.6) and (1.9) are the starting point for the arguments which lead to the Lorentz transformations. From the latter, one deduces a posteriori that Eq. (1.27) is just the Lorentz transform of the spatial coordinate of the point of S with coordinates $(x_0,\, t_u = x_0/c)$; the corresponding point of S' will have coordinates $(x'_0,\, t'_{u'} = \sqrt{1 - \nu^2/c^2}\, t_{u'})$.

The above analysis shows the important fact that the relativity of time is the result of the conspiracy of three basic facts, namely the finite velocity of light, the equivalence of the inertial observers, and the uniqueness of the clock synchronization procedure.