

## Contents

|          |   |                |
|----------|---|----------------|
|          | <i>Preface</i>  | <i>page xi</i> |
|          | <b>Introduction</b>                                   | 1              |
|          | 0.1 The leitmotiv                                     | 1              |
|          | 0.2 A few words about the history of entropy          | 3              |
|          | 0.3 Multiple meanings of entropy                      | 4              |
|          | 0.4 Conventions                                       | 18             |
|          | <br>  |                |
|          | <b>PART I Entropy in ergodic theory</b>               | 21             |
| <b>1</b> | <b>Shannon information and entropy</b>                | 23             |
|          | 1.1 Information and entropy of probability vectors    | 23             |
|          | 1.2 Partitions and sigma-algebras                     | 30             |
|          | 1.3 Information and static entropy of a partition     | 32             |
|          | 1.4 Conditional static entropy                        | 33             |
|          | 1.5 Conditional entropy via probabilistic tools*      | 35             |
|          | 1.6 Basic properties of static entropy                | 36             |
|          | 1.7 Metrics on the space of partitions                | 42             |
|          | 1.8 Mutual information*                               | 46             |
|          | 1.9 Non-Shannon inequalities*                         | 48             |
|          | Exercises   | 51             |
| <b>2</b> | <b>Dynamical entropy of a process</b>                 | 53             |
|          | 2.1 Subadditivity                                     | 53             |
|          | 2.2 Preliminaries on dynamical systems                | 57             |
|          | 2.3 Dynamical entropy of a process                    | 60             |
|          | 2.4 Properties of dynamical entropy                   | 65             |
|          | 2.5 Affinity of dynamical entropy                     | 68             |
|          | 2.6 Conditional dynamical entropy via disintegration* | 69             |

|          |   |     |
|----------|---|-----|
| viii     | <i>Contents</i>                                   |     |
|          | 2.7 Summary of the properties of entropy          | 72  |
|          | 2.8 Combinatorial entropy                         | 73  |
|          | Exercises   | 78  |
| <b>3</b> | <b>Entropy theorems in processes</b>              | 80  |
|          | 3.1 Independence and $\varepsilon$ -independence  | 80  |
|          | 3.2 The Pinsker sigma-algebra in a process        | 85  |
|          | 3.3 The Shannon–McMillan–Breiman Theorem          | 89  |
|          | 3.4 The Ornstein–Weiss Return Times Theorem       | 94  |
|          | 3.5 Horizontal data compression                   | 97  |
|          | Exercises   | 100 |
| <b>4</b> | <b>Kolmogorov–Sinai Entropy</b>                   | 102 |
|          | 4.1 Entropy of a dynamical system                 | 102 |
|          | 4.2 Generators                                    | 105 |
|          | 4.3 The natural extension                         | 111 |
|          | 4.4 Joinings                                      | 116 |
|          | 4.5 Ornstein Theory*                              | 120 |
|          | Exercises   | 130 |
| <b>5</b> | <b>The Ergodic Law of Series*</b>                 | 132 |
|          | 5.1 History of the Law of Series                  | 132 |
|          | 5.2 Attracting and repelling in signal processes  | 135 |
|          | 5.3 Decay of repelling in positive entropy        | 139 |
|          | 5.4 Typicality of attracting for long cylinders   | 152 |
|          | <b>PART II Entropy in topological dynamics</b>    | 157 |
| <b>6</b> | <b>Topological entropy</b>                        | 159 |
|          | 6.1 Three definitions of topological entropy      | 159 |
|          | 6.2 Properties of topological entropy             | 165 |
|          | 6.3 Topological conditional and tail entropies    | 167 |
|          | 6.4 Properties of topological conditional entropy | 171 |
|          | 6.5 Topological joinings                          | 172 |
|          | 6.6 The simplex of invariant measures             | 175 |
|          | 6.7 Topological fiber entropy                     | 179 |
|          | 6.8 The major Variational Principles              | 181 |
|          | 6.9 Determinism in topological systems            | 190 |
|          | 6.10 Topological preimage entropy*                | 197 |
|          | Exercises   | 199 |

| <i>Contents</i> |  | ix  |
|-----------------|--|-----|
| <b>7</b>        | <b>Dynamics in dimension zero</b>                                    | 201 |
|                 | 7.1 Zero-dimensional dynamical systems                               | 201 |
|                 | 7.2 Topological entropy in dimension zero                            | 202 |
|                 | 7.3 The invariant measures in dimension zero                         | 203 |
|                 | 7.4 The Variational Principle in dimension zero                      | 205 |
|                 | 7.5 Tail entropy and asymptotic $h$ -expansiveness in dimension zero | 206 |
|                 | 7.6 Principal zero-dimensional extensions                            | 212 |
|                 | Exercises  | 225 |
| <b>8</b>        | <b>The entropy structure</b>   | 227 |
|                 | 8.1 The type of convergence  | 227 |
|                 | 8.2 U.s.d.a.-sequences on simplices                                  | 244 |
|                 | 8.3 Entropy of a measure with respect to a topological resolution    | 254 |
|                 | 8.4 Entropy structure  | 263 |
|                 | Exercises  | 270 |
| <b>9</b>        | <b>Symbolic extensions</b>   | 272 |
|                 | 9.1 What are symbolic extensions?                                    | 272 |
|                 | 9.2 The Symbolic Extension Entropy Theorem                           | 274 |
|                 | 9.3 Properties of symbolic extension entropy                         | 287 |
|                 | 9.4 Symbolic extensions of interval maps                             | 293 |
|                 | Exercises  | 301 |
| <b>10</b>       | <b>A touch of smooth dynamics*</b>                                   | 303 |
|                 | 10.1 Margulis–Ruelle Inequality and Pesin Entropy Formula            | 303 |
|                 | 10.2 Tail entropy estimate   | 307 |
|                 | 10.3 Symbolic extensions of smooth systems                           | 308 |
|                 | <b>PART III Entropy theory for operators</b>                         | 311 |
| <b>11</b>       | <b>Measure-theoretic entropy of stochastic operators</b>             | 313 |
|                 | 11.1 A few words on operator dynamics                                | 313 |
|                 | 11.2 The axiomatic measure-theoretic definition                      | 316 |
|                 | 11.3 An explicit measure-theoretic definition                        | 329 |
|                 | 11.4 Not so bad properties of the operator entropy                   | 332 |
|                 | Exercises  | 335 |
| <b>12</b>       | <b>Topological entropy of a Markov operator</b>                      | 336 |
|                 | 12.1 Three definitions   | 336 |
|                 | 12.2 Properties of the topological operator entropy                  | 339 |

|                   |   |     |
|-------------------|---|-----|
| x                 | <i>Contents</i>                               |     |
|                   | 12.3 Half of the variational principle        | 341 |
|                   | Exercises                                     | 343 |
| <b>13</b>         | <b>Open problems in operator entropy</b>      | 344 |
|                   | 13.1 Questions on doubly stochastic operators | 344 |
|                   | 13.2 Questions concerning Markov operators    | 345 |
| <i>Appendix A</i> | <b>Toolbox</b>                                | 347 |
| <i>Appendix B</i> | <b>Conditional S–M–B</b>                      | 366 |
|                   | <i>List of symbols</i>                        | 374 |
|                   | <i>References</i>                             | 379 |
|                   | <i>Index</i>                                  | 386 |