

Entropy in Dynamical Systems

This comprehensive text on entropy covers three major types of dynamics: measure-preserving transformations; continuous maps on compact spaces; and operators on function spaces.

Part I contains proofs of the Shannon–McMillan–Breiman Theorem, the Ornstein–Weiss Return Time Theorem, the Krieger Generator Theorem, the Sinai and Ornstein Theorems, and among the newest developments, the Ergodic Law of Series. In Part II, after an expanded exposition of classical topological entropy, the book addresses Symbolic Extension Entropy. It offers deep insight into the theory of entropy structure and explains the role of zero-dimensional dynamics as a bridge between measurable and topological dynamics. Part III explains how both measure-theoretic and topological entropy can be extended to operators on relevant function spaces.

Intuitive explanations, examples, exercises and open problems make this an ideal text for a graduate course on entropy theory. More experienced researchers can also find inspiration for further research.

TOMASZ DOWNAROWICZ is Full Professor in Mathematics at Wrocław University of Technology, Poland.

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Preface

This book is designed as a comprehensive lecture on entropy in three major types of dynamics: measure-theoretic, topological and operator. In each case the study is restricted to the most classical case of the action of iterates of a single transformation (or operator) on either a standard probability space or on a compact metric space. We do not venture into studying actions of more general groups, dynamical systems on noncompact spaces or equipped with infinite measures. On the other hand, we do not restrict the generality by adding more structure to our spaces. The most structured systems addressed here in detail are smooth transformations of the compact interval. The primary intention is to create a self-contained course, from the basics through more advanced material to the newest developments. Very few theorems are quoted without a proof, mainly in the chapters or sections marked with an asterisk. These are treated as “nonmandatory” for the understanding of the rest of the book, and can be skipped if the reader chooses. Our facts are stated as generally as possible within the assumed scope, and wherever possible our proofs of classical theorems are different from those found in the most popular textbooks. Several chapters contain very recent results for which this is a textbook debut.

We assume familiarity of the reader with basics of ergodic theory, measure theory, topology and functional analysis. Nevertheless, the most useful facts are recalled either in the main text or in the appendix.

Some elementary statements and minor passages are left without a proof, as an exercise for the reader. Such statements are collected at the end of each chapter, together with other exercises of independent interest. It is planned that solutions to selected exercises will be made available shortly after the book has occurred in print, at the publisher’s website www.cambridge.org/9780521888851.

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