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Introduction

It is fair to say that the subject known today as complex dynamics – the study of iterations of analytic functions – originated in the pioneering works of P. Fatou and G. Julia early in the twentieth century (see the references [Fat] and [Ju]). In possession of what was then a new tool, Montel's theorem on normal families, Fatou and Julia each investigated the iteration of rational maps of the Riemann sphere and found that these dynamical systems had an extremely rich orbit structure. They observed that each rational map produced a dichotomy of behavior for points on the Riemann sphere. Some points – constituting a totally invariant open set known today as the *Fatou set* – showed an essentially dissipative or wandering character under iteration by the map. The remaining points formed a totally invariant compact set, today called the *Julia set*. The dynamics of a rational map on its Julia set showed a very complicated recurrent behavior, with transitive orbits and a dense subset of periodic points. Since the Julia set seemed so difficult to analyse, Fatou turned his attention to its complement (the Fatou set). The components of the Fatou set are mapped to other components, and Fatou observed that these seemed to eventually to fall into a periodic cycle of components. Unable to prove this fact, but able to verify it for many examples, Fatou nevertheless conjectured that *rational maps have no wandering domains*. He also analysed the periodic components and was essentially able to classify them into finitely many types.

It soon became apparent that even the *local* dynamics of an analytic map was not well understood. It was not always possible to linearize the dynamics of a map near a fixed or periodic point, and remarkable examples to that effect were discovered by H. Cremer. In the succeeding decades researchers in the subject turned to this linearization problem, and the more global aspects of the dynamics of rational maps were all but forgotten for about half a century.

With the arrival of fast computers and the first pictures of the Mandelbrot set, interest in the subject began to be revived. People could now draw computer pictures of Julia sets that were not only of great beauty but also inspired new conjectures. But the real revolution in the subject came with the work of D. Sullivan in the early 1980s. He was the first to realize that the Fatou–Julia theory was strongly linked to the theory of Kleinian groups, and he established a *dictionary* between the two theories. Borrowing a fundamental technique first used by Ahlfors in the theory of Kleinian groups, Sullivan proved Fatou's long-standing conjecture on wandering domains. With this theorem Sullivan started a new era in the theory of iterations of rational functions.

Our goal in this book is to present some of the main tools that are relevant to these developments (and to other more recent ones). Our efforts are concentrated on the exposition of only a few tools. We tried to select at least one interesting dynamical application for each tool presented, but it was not possible to be very systematic. Many interesting techniques had to be omitted, as well as many of the more interesting contemporary applications. There are a number of superbly written texts in complex dynamics with a more systematic exposition of theory; we strongly recommend [B2], [CG], [Mi1], [MNTU], as well as the more specialized [McM1, McM2].

The remainder of this introduction is devoted to a more careful explanation of the basic concepts involved in the above discussion and also to a brief description of the contents of the book.

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, $\mathrm{Poly}_d(\mathbb{C})$ be the space of polynomials of degree d and $\mathrm{Rat}_d(\widehat{\mathbb{C}})$ be the space of rational functions of degree d, $d \geq 2$. An element $f \in \mathrm{Rat}_d(\widehat{\mathbb{C}})$ is the quotient of two polynomials of degree $\leq d$. If the derivative of f at p vanishes or, equivalently, if f is not locally one to one in any small neighborhood of p, we say that p is a *critical* point of f and its image $f(p)$ is a *critical value* of f. When f is a polynomial of degree d, the point ∞ is always a critical point of multiplicity $d-1$ (the polynomial f is d to one in a neighborhood of ∞). The number of points in the pre-image of a point that is not a critical value is constant and equal to the degree of the rational map f. The sum of the multiplicities of critical points of a rational map of degree d is equal to $2d-2$. In particular, a polynomial of degree d has $d-1$ finite critical points. The iterates of f are the rational maps $f^1 = f$, $f^n = f \circ f^{n-1}$. The *forward* *orbit* of a point p is the subset $O^+(p) = \{f^n(p), n \geq 0\}$, its *backward orbit* is $O^-(p) = \{w \in \mathbb{C}; f^n(w) = p, n \geq 0\}$ and its *grand orbit* is $O(p) = \{w \in \widehat{\mathbb{C}}; f^n(w) = f^m(p), m, n \geq 0\}$.

Two maps f, g are *topologically conjugate* if there is a homeomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $h \circ f = g \circ h$. It follows that $h \circ f^n = g^n \circ h$ for all n and hence that the *conjugacy* h maps orbits of f into orbits of g. Since h is continuous, it preserves the asymptotic behavior of the orbits. A rational map f is *structurally stable* if there is a neighborhood of f in the space of rational maps of the same degree such that each map in this neighborhood is topologically conjugate to f.

We will consider also some special analytic families of rational maps. By such a family we mean an analytic map $F\colon \Lambda \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where Λ is an open set of parameters in some complex Banach space, such that $F_\lambda : z \mapsto F(\lambda, z)$ is a rational map for each $\lambda \in \Lambda$. The space of polynomials of degree d is an example of an analytic family of rational maps. We will also consider the notion of structural stability with respect to such a family: F_λ is structurally stable, with respect to that family, if there is a neighborhood of λ in the parameter space Λ such that, for μ in this neighborhood, F_μ is topologically conjugate to F_λ. The complement of the stable parameter values is called the *bifurcation set* of the family. It is clearly a closed subset of the parameter space Λ.

Given a rational map f, the phase space $\widehat{\mathbb{C}}$ decomposes into the disjoint union of two totally invariant subsets, the *Fatou* set $F(f)$ and the *Julia* set $J(f)$. A point z belongs to the Fatou set if there exists a neighborhood V of z such that the restrictions of all iterates f^n to this neighborhood form an equicontinuous family of functions that is, by the Arzelá–Ascoli theorem, a pre-compact family in the topology of uniform convergence on compact subsets. Therefore the Fatou set is an open set where the dynamics is simple. The Julia set, its complement, is a compact subset of the Riemann sphere. The topological and dynamical structure of these sets was the main object of study of Fatou [Fat], Julia [Ju] and others, using compactness results for families of holomorphic functions such as Montel's theorem, mentioned above, and Koebe's distortion theorem. These tools will be discussed in Chapter 3.

As mentioned earlier, a complete understanding of the structure of the Fatou set for any rational map had to wait until the 1980s, when Sullivan brought to the subject the theory of deformations of conformal structures. We will discuss this theory in Chapter 4. Sullivan proved the *no-wandering-domains* theorem [Su], which states that each connected component of the Fatou set is eventually mapped into a periodic component and that the number of periodic cycles of components is bounded. See section 4.6 for the proof of Sullivan's theorem.

The two main results of the deformation theory of conformal structures are the Ahlfors–Bers theorem, which will be discussed in section 4.4, and the theorem on the extensions and quasiconformality of holomorphic motions, which will be discussed in Chapter 5. Using these two important tools, it is proved in [MSS] and in [McS] that the set of stable parameter values is dense in any analytic family of rational maps. In particular the set of structurally stable rational maps is open and dense. See section 5.4 for a proof of this fundamental structural stability result. However, the bifurcation set is also a large and intricate set. In fact, M. Rees proved in [Re] that, in the space $\mathrm{Rat}_d(\widehat{\mathbb{C}})$ of all rational maps, the bifurcation set has positive Lebesgue measure. Also, for non-trivial analytic families of rational maps, Shishikura [Sh2] and McMullen [McM3] proved that the Hausdorff dimension of the bifurcation set is equal to the dimension of the parameter space.

A much deeper understanding of the dynamics and bifurcation patterns has been obtained for the special family of quadratic polynomials $\{f_c(z) = z^2 + c \,|\, c \in \mathbb{C}\}$. On the one hand, for values of c outside the ball of radius 2 the iterates of the critical point 0 escape to infinity, and, as we shall see in section 3.3, the Julia set is a Cantor set and all the corresponding parameter values belong to the same topological conjugacy class. On the other hand, Douady and Hubbard proved in [DH2] that the set of parameter values for which the critical orbit is bounded, the so-called *Mandelbrot set* , is connected (see also [DH1] or [MNTU], pp. 21–2, for a proof) and also showed that its interior is a countable union of disjoint topological disks. Each of these disks, with the possible exception of an interior point that corresponds to a map having a periodic critical point, is a full conjugacy class. The bifurcation set of the quadratic family is the union of the boundary of the Mandelbrot set and the countable discrete set of maps with periodic critical points that lie in the interior of the Mandelbrot set. Significant progress in the understanding of the structure of this set, as well as of the Julia sets of quadratic polynomials, has been obtained in the works of Yoccoz, McMullen, Lyubich, Graczyk-Swiatek and others.

The mathematical tools that we will discuss in this book have been also very important in the study of the dynamics of circle and interval maps that are real analytic or even smooth; see [MS], [dFM2], [dFM1], and also [dFMP]. In this case the phase space reduces to a compact interval of the real line or to the circle, but the parameter space becomes an infinite-dimensional Banach space. Holomorphic methods still play an important role in the understanding of the small-scale structure of the orbits of smooth maps.

We conclude this introduction by mentioning some fundamental open problems. The most well-known, the so-called *Fatou conjecture*, states that for a structurally stable map each critical point is in the basin of an attracting periodic point. This is a very difficult problem, which is still open even for the quadratic family. For a long time it was conjectured that the Julia set of a rational map would either be the whole Riemann sphere or would have Lebesgue measure zero; in particular, the Julia set of every polynomial would have measure zero. This was recently disproved by X. Buff and A. Cheritat [BuC], who found Julia sets of positive Lebesgue measure in the quadratic family, a truly outstanding achievement. By M. Rees' theorem, the bifurcation set of the family of rational maps of degree d has positive Lebesgue measure. However, it is expected that the bifurcation set of the family of polynomials of degree d should have zero Lebesgue measure. For the quadratic family, an important conjecture formulated by Douady and Hubbard is that the Mandelbrot set is locally connected. They proved in [DH2] that this conjecture implies Fatou's conjecture for the quadratic family. Finally, a very important open question concerns the regularity of the conjugacy between two rational maps. It is conjectured that if two rational maps are topologically conjugate then a conjugacy exists between them that is quasiconformal. A solution to this conjecture in the special case of *real* quadratic polynomials, which implies the solution of Fatou's problem, was obtained in [L4] and in [GS1] and uses all the tools that we discuss in this book and more.

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Preliminaries in complex analysis

Complex analysis is a vast and very beautiful subject, and the key to its beauty is the harmonious coexistence of analysis, algebra, geometry and topology in its most fundamental entity, the complex plane. We will assume that the reader is already familiar with the basic facts about analytic functions in one complex variable, such as Cauchy's theorem, the Cauchy–Riemann equations, power series expansions, residues and so on. Holomorphic functions in one complex variable enjoy a double life, as they can be viewed both as *analytic* objects (power series, integral representations) and as *geometric* objects (conformal mappings). The topics presented in this book exploit freely this dual character of holomorphic functions. Our purpose in this short chapter is to present some well-known or not so well-known analytic and geometric facts that will be necessary later. The reader is warned that what follows is only a brief collection of facts to be used, not a systematic exposition of the theory. For general background reading in complex analysis, see for instance [A2], [An] or [Rud].

2.1 Analytic facts

Let us start with some differential calculus of complex-valued functions defined on some domain in the complex plane (by a domain we mean as usual a non-empty, connected, open set). The two basic differential operators of complex calculus are

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \; ; \; \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \; ,$$

so that, if $f : \Omega \to \mathbb{C}$ ($\Omega \subseteq \mathbb{C}$ being a domain) is a C^1 function, then its total derivative is

$$df \ = \ \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \ = \ \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \overline{z}}\,d\overline{z}\ ,$$

where $dz = dx + i\,dy$ and $d\overline{z} = dx - i\,dy$. We often simplify the notation even further, writing $\partial f = \partial f/\partial z$ and $\overline{\partial} f = \partial f/\partial \overline{z}$ respectively. Thus, a C^1 function is analytic, or *holomorphic*, if $\overline{\partial} f(z) = 0$ for all $z \in \Omega$. In this case the limit

$$f'(z) \ = \ \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists and equals $\partial f(z)$ for all $z \in \Omega$ (it is the complex derivative of f at z).

Several basic facts from standard calculus can be restated in complex notation. Thus, we can write Green's formula in the following way. If $u, v : \Omega \to \mathbb{C}$ are C^1 functions and $V \subseteq \Omega$ is a simply connected domain bounded by a piecewise C^1 Jordan curve (the boundary ∂V), then

$$\int_{\partial V} u\,dz + v\,d\overline{z} \ = \ \iint_V \left(\partial v - \overline{\partial} u\right)\, dz \wedge d\overline{z}\,; \tag{2.1}$$

here $dz \wedge d\overline{z} = (dx + i\,dy) \wedge (dx - i\,dy) = -2i\,dx \wedge dy$ is the complex area form.

With the help of Green's formula, it is not difficult to check that if $f : \Omega \to \mathbb{C}$ is a C^1 function and D is an open disk with $\overline{D} \subseteq \Omega$ then for all $z \in D$ we have

$$f(z) \ = \ \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\iint_D \frac{\overline{\partial} f(\zeta)}{\zeta - z}\,d\zeta \wedge d\overline{\zeta}\,. \tag{2.2}$$

This is known as the *Cauchy–Green* or *Pompeiu* formula Note that if f is analytic then the second integral vanishes identically and we recover the usual Cauchy formula of basic complex analysis.

We can still make sense out of the preceding formulas even if the functions involved are not C^1. Indeed, we can think of ∂f or $\overline{\partial} f$ as *distributions*. The most useful situation occurs when the distributional derivatives $\partial f, \overline{\partial} f$ of a given $f : \Omega \to \mathbb{C}$ are represented by locally integrable functions $f_z, f_{\overline{z}} : \Omega \to \mathbb{C}$. In this case, for all test functions $\varphi \in C_0^\infty(\Omega)$ (complex-valued C^∞ functions with compact support) we have

$$\iint_\Omega f_z(\zeta)\varphi(\zeta)\,d\zeta \wedge d\overline{\zeta} \ = \ -\iint_\Omega f(\zeta)\partial\varphi(\zeta)\,d\zeta \wedge d\overline{\zeta}\ ,$$

as well as

$$\iint_\Omega f_{\overline{z}}(\zeta)\varphi(\zeta)\, d\zeta \wedge d\overline{\zeta} \;=\; -\iint_\Omega f(\zeta)\overline{\partial}\varphi(\zeta)\, d\zeta \wedge d\overline{\zeta}.$$

When working with the distributional derivatives of a given f, as above, we often need to approximate f in a suitable sense by a sequence of smooth functions usually referred to as a *smoothing sequence*. Such an approximation is obtained by performing the convolution of f with an *approximate identity*, a sequence of C^∞ functions $\phi_n : \mathbb{C} \to \mathbb{C}$ with compact support having the following properties:

(1) $\iint_{\mathbb{C}} |\phi_n(z)|\, dxdy = 1$
(2) $\operatorname{supp}\phi_n \subset D(0, 1/n)$.

The standard example of an approximate identity is constructed as follows. Let $\varphi : \mathbb{C} \to \mathbb{R}$ be the C^∞ function which is given by

$$\varphi(z) \;=\; \exp\left(-\frac{1}{1-|z|^2}\right)$$

for all $z \in \mathbb{D}$ and which vanishes identically outside $\mathbb{D}$. Let $\phi : \mathbb{C} \to \mathbb{R}$ be defined by $\phi(z) = \varphi(z)/\iint_{\mathbb{C}} |\varphi(z)|\, dxdy$, and then take $\phi_n(z) = n^2\phi(nz)$ for each $n \geq 1$. Now we have the following important fact.

Lemma 2.1.1 *Let $f : \Omega \to \mathbb{C}$ be a continuous function whose distributional derivatives $f_z, f_{\overline{z}}$ are such that $|f_z|^p$ and $|f_{\overline{z}}|^p$ are locally integrable on Ω, for some fixed $p \geq 1$. Then for each compact set $K \subset \Omega$ there exists a sequence $f_n \in C_0^\infty(\Omega)$ such that f_n converges uniformly to f on K as $n \to \infty$ and such that*

$$\lim_{n\to\infty} \iint_K |\partial f_n(z) - f_z(z)|^p\, dxdy \;=\; 0$$

as well as

$$\lim_{n\to\infty} \iint_K |\overline{\partial} f_n(z) - f_{\overline{z}}(z)|^p\, dxdy \;=\; 0\ .$$

Such a sequence is called an L^p smoothing sequence for f in K.

Proof Consider a fixed $\lambda_K \in C_0^\infty(\Omega)$ that is constant and equal to 1 on some neighborhood of K, and form the function $f_K = \lambda_K f$, which has compact support in K; extend f_K outside Ω, setting it equal to zero.

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Note that the distributional partial derivatives of f_K are locally in L^p; in fact

$$\partial f_K = \partial\lambda_K f + \lambda_K \partial f \ , \ \overline{\partial} f_K = \overline{\partial}\lambda_K f + \lambda_K \overline{\partial} f \ .$$

Let $\{\phi_n\}_{n\geq 1}$ be an approximate identity (say the one we constructed before lemma 2.1.1), and let $f_n \in C_0^\infty(\Omega)$ be given by

$$f_n(z) = \phi_n * f_K(z) = -\frac{1}{2i}\iint_{\mathbb{C}} \phi_n(z-\zeta) f_K(\zeta)\, d\zeta \wedge d\overline{\zeta} \ .$$

Then we have also

$$\partial f_n = \phi_n * \partial f_K \ , \qquad \overline{\partial} f_n = \phi_n * \overline{\partial} f_K \ .$$

It now follows from standard properties of convolutions that $f_n \to f_K$ uniformly in K and that $\partial f_n \to \partial f_K$ and $\overline{\partial} f_n \to \overline{\partial} f_K$ in $L^p(K)$. This is the desired result, because for all $z \in K$ we have $f_K(z) = f(z)$, $\partial f_K(z) = f_z(z)$ and $\overline{\partial} f_K(z) = f_{\overline{z}}(z)$. □

This lemma yields two key results. The first is a fundamental lemma due to H. Weyl, which is a special case of a much more general regularity theorem for elliptic operators.

Proposition 2.1.2 (Weyl's lemma) *If $f : \Omega \to \mathbb{C}$ is a continuous function such that $\overline{\partial} f = 0$ in Ω in the sense of distributions then f is holomorphic in Ω.*

Proof Take any disk D whose closure is contained in Ω. Let $f_n : \Omega \to \mathbb{C}$ be an L^1 smoothing sequence for f in $\overline{D}$. Then f_n converges uniformly to f in $\overline{D}$. From the fact that $\overline{\partial} f = 0$ in the distributional sense, it follows that $\overline{\partial} f_n(z) = 0$ for all $z \in D$. Since f_n is C^1, it follows that f_n is holomorphic for each n. Therefore f, being the uniform limit of holomorphic functions, is holomorphic in D. Since $D \subset \Omega$ is arbitrary, f is holomorphic in Ω. □

The second result that we prove with the help of lemma 2.1.1 is the following more general version of (2.2).

Proposition 2.1.3 (Pompeiu's formula) *Let $f : \Omega \to \mathbb{C}$ be a continuous function whose distributional derivatives $\partial f, \overline{\partial} f$ are represented by functions $f_z, f_{\overline{z}}$ locally in L^p for some fixed p with $2 < p < \infty$. Then for each open disk D compactly contained in Ω and each $z \in D$ we have*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i}\iint_D \frac{f_{\overline{z}}(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta} \ . \qquad (2.3)$$

Proof Note that the last integral is absolutely convergent because $f_{\overline{z}} \in L^p(D)$ and $1/(\zeta - z) \in L^q(D)$, where $q < 2$ is the conjugate exponent of p (that is, $p^{-1} + q^{-1} = 1$). Let $f_n \in C_0^\infty(\Omega)$ be an L^p smoothing sequence for f in D. By (2.2), for each n we have

$$f_n(z) \;=\; \frac{1}{2\pi i}\int_{\partial D} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i}\iint_D \frac{\overline{\partial} f_n(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}\ .$$

Since $f_n \to f$ uniformly in $\overline{D}$, whereas $\overline{\partial} f_n \to f_{\overline{z}}$ in $L^p(\overline{D})$, we deduce (say by the dominated convergence theorem) that (2.3) holds. □

2.2 Geometric inequalities

The theory of conformal mappings is extremely rich in inequalities having a geometric content.

2.2.1 The classical Schwarz lemma

The most fundamental inequality in complex function theory is the classical *Schwarz lemma*, which we now recall. It states that every holomorphic self-map of the unit disk that fixes the origin is either a contraction near the origin or else it is a rotation. The precise statement is the following.

Lemma 2.2.1 (Schwarz) *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map such that $f(0) = 0$, and let $\lambda = f'(0)$. Then either $|\lambda| < 1$, in which case $|f(z)| < |z|$ for all z, or else $|\lambda| = 1$, in which case $f(z) = \lambda z$ for all z.*

Proof Let $\varphi : \mathbb{D} \to \mathbb{D}$ be given by

$$\varphi(z) \;=\; \begin{cases} \dfrac{f(z)}{z} & \text{if } \ z \in \mathbb{D}\setminus\{0\}\,, \\[2ex] f'(0) & \text{if } \ z = 0\ . \end{cases}$$

By Riemann's removable singularity theorem, φ is holomorphic. Hence, for all $z \in \mathbb{D}$ and all r such that $|z| \le r < 1$ we have, by the maximum principle,

$$|\varphi(z)| \;\le\; \sup_{|\zeta|=r} \left|\frac{f(\zeta)}{\zeta}\right| \;\le\; \frac{1}{r}\ .$$

Letting $r \to 1$, we deduce that $|\varphi(z)| \le 1$, i.e. $|f(z)| \le |z|$, for all $z \in \mathbb{D}$. If equality holds for some z then, again by the maximum principle, φ