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# GALOIS GROUPS AND FUNDAMENTAL GROUPS

Ever since the concepts of Galois groups in algebra and fundamental groups in topology emerged during the nineteenth century, mathematicians have known of the strong analogies between the two concepts. This book presents the connection starting at an elementary level, showing how the judicious use of algebraic geometry gives access to the powerful interplay between algebra and topology that underpins much modern research in geometry and number theory.

Assuming as little technical background as possible, the book starts with basic algebraic and topological concepts, but already presented from the modern viewpoint advocated by Grothendieck. This enables a systematic yet accessible development of the theories of fundamental groups of algebraic curves, fundamental groups of schemes, and Tannakian fundamental groups. The connection between fundamental groups and linear differential equations is also developed at increasing levels of generality. Key applications and recent results, for example on the inverse Galois problem, are given throughout.

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# Galois Groups and Fundamental Groups

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# *Preface*

Ever since the concepts of the Galois group and the fundamental group emerged in the course of the nineteenth century, mathematicians have been aware of the strong analogies between the two notions. In its early formulation Galois theory studied the effect of substitutions on roots of a polynomial equation; in the language of group theory this is a permutation action. On the other hand, the fundamental group made a first, if somewhat disguised, appearance in the study of solutions of differential equations in a complex domain. Given a local solution of the equation in the neighbourhood of a base point, one obtains another solution by analytic continuation along a closed loop: this is the monodromy action.

Leaving the naïve idea of substituting solutions, the next important observation is that the actions in question come from automorphisms of objects that do not depend on the equations any more but only on the base. In the context of Galois theory the automorphisms are those of a separable closure of the base field from which the coefficients of the equation are taken. For differential equations the analogous role is played by a universal cover of the base domain. The local solutions, which may be regarded as multi-valued functions in the neighbourhood of a base point, pull back to single-valued functions on the universal cover, and the monodromy action is the effect of composing with its topological automorphisms.

In fact, the two situations are not only parallel but closely interrelated. If our complex domain $D$ is just the complex plane minus finitely many points, a local solution of a linear holomorphic differential equation that becomes single-valued on a cover with *finite* fibres lies in a finite algebraic extension of the field $\mathbf{C}(t)$ of meromorphic functions on $D$. We actually obtain a one-to-one correspondence between finite extensions of $\mathbf{C}(t)$ and finite covers of $D$, provided that we allow finitely many exceptional points called branch points. This opens the way for developing a unified theory within the category of algebraic curves, first over the complex numbers and then over a general base field. Algebraic geometers working in the 1950s realized that the theory generalizes without much effort to higher dimensional algebraic varieties satisfying the *normality* condition.

A further step towards generality was taken by Grothendieck. He gave a definition of the fundamental group not as the automorphism group of some space or field but as the automorphism group of a *functor*, namely the one that assigns to a cover its fibre over the fixed base point. This point of view permits a great clarification of earlier concepts on the one hand, and the most general definition of the fundamental group in algebraic geometry on the other. One should by no means regard it as mere abstraction: without working in the general setting many important theorems about curves could not have been obtained.

Grothendieck's concept of the algebraic fundamental group gives a satisfactory theory as far as finite covers of algebraic varieties or schemes are concerned, but important aspects of previous theories are lost because one restricts attention to finite covers. According to the fruitful motivic philosophy of Grothendieck and Deligne that underlies much of current research, this can be remedied by considering the algebraic fundamental group as only one incarnation, the 'étale realization' of a more general object. Other incarnations are the 'topological realization' where not necessarily finite covers of topological spaces are brought into play, and the 'de Rham realization' which is an algebraic formalization of the theory of differential equations. For the definition of the latter Grothendieck envisioned the algebraic formalism of Tannakian categories worked out in detail by Saavedra and Deligne. The various realizations of the fundamental group are related by comparison theorems. One instance of these is the correspondence between covers and field extensions mentioned above. Another one is the Riemann–Hilbert correspondence relating linear differential equations to representations of the fundamental group.

We have to stop here, as we have reached the viewpoint of present-day algebraic geometers and algebraic analysts on the subject. The future may well bring further unifications highlighting hitherto neglected aspects. Still, we feel that the time is ripe for a systematic discussion of the topic starting from the basics, and this is the aim of the present book. A glance at the table of contents shows that we shall be following the line of thought sketched above. Along the way we shall also mention a number of applications and recent results.

The first three chapters may be read by anyone acquainted with basic field theory, point set topology and the rudiments of complex analysis. Chapter 4 treats algebraic geometry, but is meant to be accessible to readers with no previous knowledge of the subject; the experts will skip a few introductory sections. The last two chapters are of a slightly more advanced nature. Nevertheless, we give a detailed summary of the basics on schemes at the beginning of Chapter 5, while most of Chapter 6 assumes only basic algebra and is largely independent of previous chapters.

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