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Review of Analytic Function Theory

1.1 Preamble

Complex analysis is the foundation for everything in this book. Special functions, integral transforms, Green's functions, orthogonal function expansions, and classical asymptotic techniques like steepest descent cannot be properly understood or used without a thorough understanding of analytic function theory. We provide here only a review; the student for whom this is a first exposure to the subject ought to consult other texts that treat these topics exclusively. There are a vast number of such books – many of them are very helpful. Among them, we cite a very complete book for applied mathematicians, engineers, and scientists by Carrier, Krook, and Pearson and the exhaustive treatise by Markushevich.

Because the subject was given birth by the need to solve problems in fluid dynamics and electromagnetism, there is also a significant library of books on those topics that make intensive use of complex-variable methods. In the area with which the authors are most familiar, the classic book on hydrodynamics by Milne-Thomson is a great resource. In particular, there is much attention paid there to conformal mappings – a topic not discussed here because it is not directly helpful in most solution methods for partial differential equations – with some obvious exceptions.

1.2 Fundamentals of Complex Numbers

A complex number, c, is defined by

$$c = a + ib, \quad i \equiv \sqrt{-1}, \tag{1.1}$$

where a and b are real numbers. The quantities a and b are called, respectively, the "real" and "imaginary" parts of the complex number, c. Thus, we will write $a = \Re\text{eal}(c)$ and $b = \Im\text{mag}(c)$. It is often convenient to work with the *complex conjugate* of the number, c, which we will denote always in this book by the notation $\bar{c}$. (In some books, the notation c^* is used.) The complex conjugate of c is obtained by

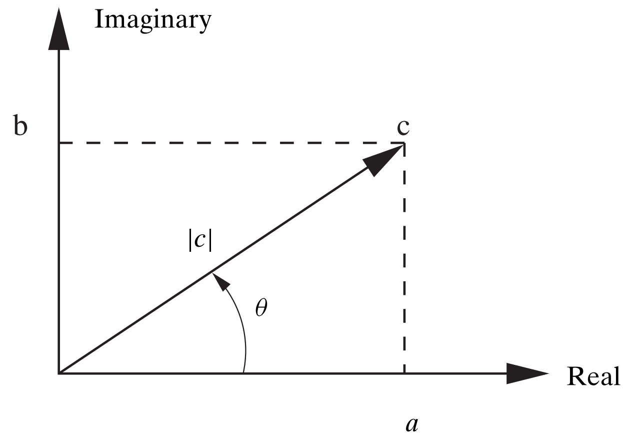

Figure 1.1. Argand diagram for a complex number.

everywhere replacing i by $(-i)$. Hence, $\bar{c} = a - ib$. The *modulus* of the complex number is a norm, defined by $|c| = \sqrt{a^2 + b^2}$. Note, then, that

$$|c|^2 = a^2 + b^2 = c\bar{c} = \bar{c}c.$$

Complex numbers may also then be written in *polar* form, so that

$$c = |c| \cos\theta + i|c| \sin\theta = |c|e^{i\theta},$$

with the angle as indicated in Figure 1.1. This angle θ will be called the "argument" of z. Hence, $\theta = \arg(z)$.

Any complex number can be displayed in a plane of numbers by specifying either (a, b) or $(|c|, \theta)$. The horizontal axis is the axis of *real parts*, and the vertical axis is for the *imaginary parts*. The conjugate of any complex number is then located at its reflection in the horizontal axis.

For doing arithmetic with these numbers, clearly addition and multiplication result in

$$c_1 + c_2 = (a_1 + a_2) + i(b_1 + b_2),$$
$$c_1 \times c_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1).$$

If a complex number is written as the sum of a real part and i times its imaginary part, then division can be done by multiplying numerator and denominator by the complex conjugate of the denominator. Consider the following example:

$$\frac{1+2i}{2-i} = \frac{1+2i}{2-i}\frac{2+i}{2+i} = \frac{5i}{2^2+1^2} = i.$$

1.2.1 Complex Roots and Logarithms

Before turning to more advanced questions of differentiation, integration, and so on, we note something quite useful about roots that is mysterious with real quantities but makes sense in the complex plane. Suppose we wish to find the cube root of 8. The real root, the one the calculator gives you, or you have memorized, is 2. However, according to the fundamental theorem of algebra, there are *three* cube roots of 8. We can find them as follows:

$$z = 8^{\frac{1}{3}} = \left(8e^{2n\pi i}\right)^{\frac{1}{3}}. \tag{1.2}$$

The quantity n is an integer. Since

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

we see that $\exp(2n\pi i) = \sin(2n\pi) + i\sin(2n\pi) \equiv 1$. Continuing with what we wrote in (1.2), we have

$$z = 8^{\frac{1}{3}} = \left(8e^{2n\pi i}\right)^{\frac{1}{3}} = 2e^{2n\pi i/3}.$$

Therefore, there are *three* roots, spaced about the origin 120 degrees apart.

Consider, as a second example, the roots of the polynomial

$$\lambda^4 + 16 = 0. \tag{1.3}$$

Rearranging,

$$\lambda = (-16)^{\frac{1}{4}} = \left(16e^{(2n\pi+\pi)i}\right)^{\frac{1}{4}} = 2e^{i\pi/4+n\pi i/2}.$$

So, the four roots of the quartic equation all have modulus 2 and are spaced around the origin 90 degrees apart, with the $n = 0$ one located at $z = \sqrt{2}(1+i)$.

The exponential function e^z is extended to complex numbers as

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y),$$

so that Euler's identity is still valid. Notice that $e^z = 1$, if $z = 2n\pi i$, and n is any integer. Thus, e^z is periodic with period $2\pi i$,

$$e^{z+2\pi i} = e^z.$$

The logarithm is the inverse of the exponential function, defined as the "multivalued function":

$$\log z = \log|z| + i\arg z + 2\pi n i, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Many authors use $\ln|z|$ as an equivalent name for $\log|z|$. In any case, the *principal branch of the logarithm* is defined by taking $n = 0$ and $-\pi < \arg z < \pi$. (Sometimes, it may be more suitable to take the principal branch to be given by $0 < \arg z < 2\pi$.) Thus, for example,

$$\log i = \log|i| + i\arg i = \log 1 + i\pi/2 = \pi i/2,$$

with either choice of principal branch, while

$$\log(-1) = \log 1 + i\arg(-1) = \pi i,$$

with the second choice.

Fractional powers are defined through the logarithm

$$z^\alpha = e^{\alpha \log z}, \quad z \neq 0.$$

So z^α is a multivalued function when α is not an integer. For example,

$$i^i = e^{i \log i} = e^{i(\log 1 + i \arg i + 2n\pi i)}, \quad n = 0, 1, 2, \ldots$$

For its principal value, take $n = 0$ and $i^i = e^{-\pi/2}$. Much of this type of analysis extends to the inverse trigonometric and inverse hyperbolic functions. For example, since

$$\sin w = \frac{e^{iw} - e^{-iw}}{2i} = z,$$

say, then

$$w = \sin^{-1} z = \frac{1}{i} \log\left[iz + (1 - z^2)^{1/2}\right].$$

If, $z \neq \pm 1$, $(1 - z^2)^{1/2}$ has two possible values. In particular, to evaluate $\sin^{-1}(2)$, we have

$$\begin{aligned}
\sin^{-1}(2) &= \frac{1}{i} \log\left[2i \pm i\sqrt{3}\right] \\
&= \begin{cases} \frac{1}{i}\left(\log(2 - \sqrt{3}) + i\left(\frac{\pi}{2} + 2n\pi\right)\right) \\ \frac{1}{i}\left(\log(2 + \sqrt{3}) + i\left(\frac{\pi}{2} + 2n\pi\right)\right) \end{cases}, \quad n = 0, \pm 1, \pm 2, \ldots \\
&= \frac{\pi}{2} + 2n\pi - i \log(2 - \sqrt{3}),\ \frac{\pi}{2} + 2n\pi + i \log(2 + \sqrt{3}), \quad n = 0, \pm 1, \pm 2, \ldots.
\end{aligned}$$

Now, observe that since $(2 + \sqrt{3})(2 - \sqrt{3}) = 1$ and $\log(2 + \sqrt{3}) = -\log(2 - \sqrt{3})$, therefore

$$\sin^{-1}(2) = \frac{\pi}{2} + 2n\pi \pm i \log(2 + \sqrt{3}), \quad n = 0, \pm 1, \pm 2, \ldots.$$

1.3 Analytic Functions

Just as the function $y = f(x)$ assigns one real number, y, to another real number, x, so we can define functions of a complex variable, say, $z = x + iy$. Then, one analogy that might be drawn is to a vector function of two variables (x, y). Hence, for example, the function $f(z) = z^2$ takes the value 1 for $z = 1$, the value -4 for $z = 2i$, and the value $(-3 + 4i)$ for $z = 1 + 2i$. The independent variable is defined over a plane like that shown in Figure 1.1. As in the case of any complex number, this function can be split into real and imaginary parts, so that we could write the function in terms of two real functions of real variables; that is,

$$f = U(x, y) + iV(x, y), \quad z = x + iy. \tag{1.4}$$

1.3.1 Limits and Continuity

The theory of complex functions is so closely tied to that of functions of two real variables, and functionally similar it turns out, to that of a single real variable, that many of the underlying processes are often taken for granted. It is important to note that a sequence of points $z_n \to z_0$, if and only if, $(x_n + i y_n) \to (x_0 + i y_0)$. For a function $f(z)$, the process $\lim_{z\to z_0} f(z)$ takes place as

$$\lim_{(x,y)\to(x_0,y_0)} [U(x, y) + i V(x, y)] = \lim_{(x,y)\to(x_0,y_0)} U(x, y) + i \lim_{(x,y)\to(x_0,y_0)} V(x, y).$$

Thus, the definition of *continuity*,

$$\lim_{z\to z_0} f(z) = f(z_0),$$

means that

$$\lim_{(x,y)\to(x_0,y_0)} [U(x, y) + i V(x, y)] = U(x_0, y_0) + i V(x_0, y_0),$$

as well. In particular, this means that the indicated limits must be independent of the direction in which they are taken.

So, for elementary functions, such as polynomials, the limits are as to be expected:

$$\lim_{z\to 3i} (z^2 + 9) = 0.$$

Even for quotients, one may conclude that

$$\begin{aligned}\lim_{z\to 3i} \frac{z^2 + 9}{z - 3i} &= \lim_{z\to 3i} \frac{(z + 3i)(z - 3i)}{z - 3i}\\ &= \lim_{z\to 3i} (z + 3i) = 6i.\end{aligned}$$

Continuous functions are defined by their limits so $f(z) = |z|$ is everywhere continuous. However, $f(z) = \arg z$ is discontinuous at each point on the nonpositive real axis, with a cut on the non-positive real axis.

1.3.2 Differentiation

Let us consider the derivative of this function of the complex variable, z, by analogy with real variables. Then,

$$\begin{aligned}\frac{df}{dz} &= \lim_{\Delta z\to 0} \left[\frac{f(z + \Delta z) - f(z)}{\Delta z}\right] \qquad (1.5)\\ &= \lim_{\Delta z\to 0} \left[\frac{U(x + \Delta x, y + \Delta y) + i V(x + \Delta x, y + \Delta y) - U(x, y) - i V(x, y)}{\Delta x + i\Delta y}\right]\\ &= \lim_{\Delta z\to 0} \left[\frac{U(x + \Delta x, y + \Delta y) - U(x, y) + i\,(V(x + \Delta x, y + \Delta y) - V(x, y))}{\Delta x + i\Delta y}\right],\end{aligned}$$

after collecting the real and imaginary parts. The partial derivatives U_x and V_x are just those when y is held constant; $\Delta y = 0$. Likewise, U_y and V_y arise when x is held constant; $\Delta x = 0$. The result is that the derivative df/dz may be written in two ways:

$$\frac{df}{dz} = \begin{cases} U_x + iV_x, & \text{if } \Delta y = 0 \\ V_y - iU_y, & \text{if } \Delta x = 0 \end{cases}. \tag{1.6}$$

Using Taylor's theorem for a function of two variables, the expression (1.5) can be simplified. That is, the limit may be taken along some line in the complex plane along which both $\Delta x \to 0$ and $\Delta y \to 0$. The difference quotient then becomes

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \left[\frac{(U_x + iV_x)\Delta x + i(V_y - iU_y)\Delta y + \text{higher-order terms}}{\Delta x + i\Delta y} \right],$$

which may be simplified and the limit performed. The result is

$$\frac{df}{dz} = \frac{(U_x + iV_x) + iS(V_y - iU_y)}{1 + iS}, \tag{1.7}$$

where $S \equiv \Delta y/\Delta x$ is the slope of the line in the complex z-plane along which the limit is performed. The cases where $S = 0$ and $S = \infty$ were obtained in (1.6). Taking a cue from several real variables, it would be sensible if the derivative of a complex function at any location z were independent of the direction of approach to that point – a kind of generalization of the idea of continuity for real functions. Thus, we give the following definition:

Definition 1.1. *A function f, of a complex variable, z, is "analytic" in a region $\mathcal{R}$ of the complex plane if its derivative exists and is single-valued in that region.*

That means, from the above results, that f' should be independent of the constant S, and inspection shows that such independence can be achieved if and only if $U_x + iV_x = V_y - iU_y$. On equating real and imaginary parts, we obtain the Cauchy–Riemann equations, and the associated theorem is as follows in Theorem 1.1.

Theorem 1.1. *A necessary and sufficient condition for a function, $f = U + iV$, of a complex variable z to be analytic in a region $\mathcal{R}$ of the complex plane is that*

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y},$$
$$\frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}. \tag{1.8}$$

everywhere in $\mathcal{R}$, and U_x, U_y, V_x and V_y exist in $\mathcal{R}$.

Equations (1.8) are known as the Cauchy–Riemann equations.

Going back to (1.7), and inserting the Cauchy–Riemann equations, we regain the formulas for the derivative – when it exists, namely,

$$\frac{df}{dz} = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} - i\frac{\partial U}{\partial y}.$$

Note that since $z = x + iy$ and $\bar{z} = x - iy$, these equations may be inverted, so that $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/(2i)$. Then, from (1.4), in general, f is a function of both z and $\bar{z}$. However, it can easily be shown, that the derivative cannot be independent of S unless f is independent of $\bar{z}$. Thus, we have the result (see Exercise 1.3)

Corollary 1.1. *A necessary condition for a function of a complex variable to be analytic anywhere in the complex plane is that* $\partial f/\partial \bar{z} \equiv 0$ *in the plane.*

Let's put it a different way: Suppose we choose any two functions U and V of x and y. Then, we can create, from those functions, a function of a complex variable by writing $f = U + iV$ as in (1.4). Then, as noted above, this function will in general be a function of both z and $\bar{z}$. What Corollary 1.1 says is that if $\bar{z}$ appears in the expression for f, it *cannot* be an analytic function; it is impossible. (A necessary condition!) If f depends on z alone, then the function *may* be analytic in some portion of the complex z-plane.

Now, for some examples. Consider the function $f = x^2 - y^2 + 2ixy$. The reader can easily verify that the Cauchy–Riemann equations are indeed satisfied. The partial derivatives all exist in the *finite* z-plane, so we conclude that this function is analytic in the finite z-plane. Such a function is said to be an "entire function." Note, in terms of the corollary noted above, that this function can also be written as $f = z^2$. As a second example, consider $f = x^2 - y^2 - 2ixy$. The Cauchy–Riemann equations (1.8) are NOT satisfied for any z. Hence, the function is not analytic anywhere in the plane. (In fact, this function is $f = \bar{z}^2$ and hence cannot be analytic!) As a final example, consider the function $f = 1/z$. The real and imaginary parts are given by $U = x/(x^2 + y^2)$ and $V = -y/(x^2 + y^2)$. Again, the Cauchy–Riemann equations are satisfied in the plane, but note that the partial derivatives are unbounded at the origin. Hence, this function $f = 1/z$ is analytic everywhere except at $z = 0$. The origin, in this case, is a *singularity* of the function.

1.3.3 Harmonic Functions

For an analytic function $f = U + iV$, note that

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 V}{\partial x \partial y} = -\frac{\partial^2 U}{\partial y^2} \Longrightarrow \nabla^2 U = 0.$$

Both Cauchy–Riemann equations (1.8) were used in this derivation. In the same way, V can also be shown to be a solution of Laplace's equation. In this derivation, it was tacitly assumed that if f is analytic, so is f'. This is true and will be demonstrated later in Section 1.4.

At a point $z_0 = x_0 + iy_0$, where $f'(z_0) = 0$, $\nabla U = 0$ and $\nabla V = 0$, so the point (x_0, y_0) is a critical point for both of the conjugate functions. Furthermore, since U and V are harmonic, the critical point will be a saddlepoint as long as f is not a constant.

The real and imaginary parts of polynomials are thereby harmonic functions. In a similar manner, with restrictions on the domains, other harmonic functions may be constructed. For example,

$$\log z = \log \sqrt{x^2 + y^2} + i \arctan\left(\frac{y}{x}\right),$$

has harmonic components on the whole plane, except at the origin.

1.3.4 Note on Fluid Dynamics

In incompressible fluid flow, we know that both the velocity potential, ϕ, and the stream function, ψ, obey Laplace's equation. So, we can build an analytic function $F = \phi + i\psi$ to describe the fluid flow. Note that Equations (1.8) are, with these functions,

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y},$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Therefore, instead of working with real functions ϕ and ψ, we can deal with a *complex potential* F. From this derivative formula, the fluid velocity components are simply related to the derivative of F, that is,

$$F' = \phi_x + i\psi_x = u - iv.$$

So, for example, stagnation points in a flow are found by putting $F' = 0$.

A thorough development of these ideas can be found in may places; a good graduate-level reference is a book by Milne-Thomson.

1.4 Integration and Cauchy's Theorem

Consider a complex function, $f(z)$. Along some curve C from point a to point b in the plane, choose any set of $(N+1)$ points $\{z_k\}$, separated by intervals $\{\Delta z_k\}$, where $\Delta z_k = z_{k+1} - z_k$, along a segment of the curve from $a = z_0$ to $z_N = b$, such that

$$\sup_N \sum_{k=0}^{N} |\Delta z_k|$$

is finite. The curve, is called a "rectifiable curve." Consider the sum

$$S_N \equiv \sum_{k=0}^{N} f(z_k)\Delta z_k. \tag{1.9}$$

In order for the terms in this sum to have meaning, we assume f to be continuous along the curve C. If we take the limit as $N \to \infty$, define the limit of (1.9) as an integral (Markushevich, Vol. I),

$$S_\infty = \int_a^b f(z)dz. \tag{1.10}$$

This definition of the integral is similar to that of a line integral in the plane. The ideas of line integration are often used in the evaluation of complex integrals. There is an additional potential tool in the Cauchy–Riemann equations. However, to employ the Cauchy–Riemann equations on a region $\mathcal{R}$ containing C, f should be analytic. In particular, it may be shown that this integral is independent of the path chosen between a and b, if f obeys the Cauchy–Riemann equations along C, that is, if the function is analytic along C. The region $\mathcal{R}$, whether bounded or unbounded must be *simply connected*, that is, whenever $\mathcal{R}$ contains a simple, rectifiable, closed curve C (*Jordan curve*), it also contains the interior of C.

It may also be demonstrated that the familiar antiderivative forms for standard functions of real variables remain valid for antiderivatives of analytic functions, so that the antiderivative of $\exp(kz)$, for example, is $\exp(kz)/k$.

It follows that *Cauchy's theorem* is crucial to being able to do integration in the complex plane. The theorem is stated as follows in Theorem 1.2.

Theorem 1.2. *Let $\mathcal{R}$ be a simply connected region. If a function f, is single-valued and analytic on and inside a closed, rectifiable path C in $\mathcal{R}$, then $\oint_C f(z)dz = 0$.*

A simple proof involving only Green's theorem in the plane is as follows:

Write the analytic function f as

$$f(z) = u(x, y) + iv(x, y).$$

Call the region inside C as D. Then using the ideas of line integration underlying Equation (1.10), we take $dz = dx + idy$ so that

$$\oint_C f(z)dz = \oint_C (udx - vdy) + i \oint_C (vdx + udy)$$
$$= -\iint_D \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dxdy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy.$$

By application of Green's theorem and the Cauchy–Riemann equations, each of the last two integrals is zero and the result has been proved. The proof may be found in standard complex-variable texts (Markushevich, Vol. I).

Consider the integral of the function z^n, with n an integer, carried out around a circle of radius r, centered at the origin, and denoted here by C. Hence,

$$\oint_C z^n dz = ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta. \tag{1.11}$$

The Cauchy theorem states only that this integral must be zero for $n \geq 0$. Clearly, on doing the integration, we see that the integral is exactly zero for all values of n different from -1. For $n = -1$, however, the value of the integral is $2\pi i$. Therefore, we can write

$$\oint_C z^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}. \tag{1.12}$$

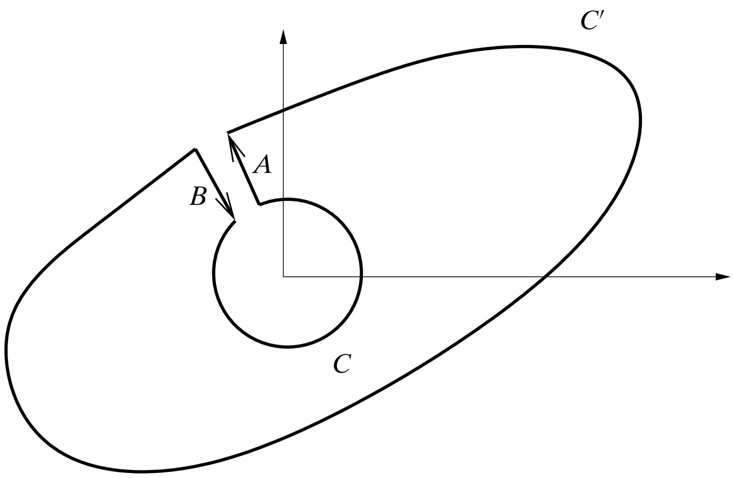

Figure 1.2. Integration path.

This result is actually more general than it appears. Consider another curve, say C', which is of arbitrary shape, which also encircles the origin and wholly contains C – f is analytic everywhere on C' and in the region between C and C'. As shown in Figure 1.2, consider now a new closed curve that is constructed by using C and C' except at a small segment of each, where the two curves are connected with two line segments A and B. Around a new closed curve formed going around C to A, along A to C', around C' in the clockwise direction, then back to C along B, the integral is zero, according to the Cauchy theorem. Since f is analytic between C and C', as we let A approach B, the integrals along those two paths just cancel, leaving the integrals along C and C' to add to zero. If we change the direction of integration along C' to the usual, positive (counterclockwise) direction, then we find that the integral around C' is exactly equal to the integral along C. Thus, the result in Equation (1.12) is independent of the shape of the path.

This discussion shows how Cauchy's theorem may be extended to regions with "holes," which are *multiply connected* regions.

The question naturally arises, does the condition $\oint_C f(z)dz = 0$ imply that f is analytic? The answer to this is known as **Morera's Theorem** and is simply

Theorem 1.3. *Let $\mathcal{R}$ be a open connected region. If a function f is continuous in $\mathcal{R}$ and $\oint_C f(z)dz = 0$ for all closed curves C in $\mathcal{R}$, then $f(z)$ is analytic in $\mathcal{R}$.*

The proof will not be presented here, but does depend on showing that such an $f(z)$ is the derivative of analytic function $F(z)$ and hence is analytic.

1.4.1 Cauchy's Integral Formula

The expression given by (1.12) is indicative of the importance of a more general line integral for analytic functions. This result, called "Cauchy's integral formula," is stated as