Part I

Physical Processes and Numerical Methods Common to Structure Formations in Astrophysics

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The physics of turbulence

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Turbulence in fluids is a topic of great interest. First and foremost, most flows in nature are turbulent and this is particularly true in the astrophysical context (Kritsuk & Norman 2004). Also, turbulence leads to very peculiar mechanics that still escapes to a great extent from our understanding. Since the pioneering works conducted by Osborne Reynolds at the end of the nineteenth century (around 1895), turbulence in fluids has become a rich and challenging research subject in which scientists from engineering, theoretical and experimental physics have been involved with many different perspectives. There is no doubt that bridging ideas from one field to another, and therefore stimulating new interdisciplinary approaches, should provide a fruitful means of gaining understanding on turbulence in the future.

In this chapter, the *background* physics of turbulence will be discussed spontaneously at a (very) basic level, i.e. without getting into details or precise formulation. The discussion will be limited to incompressible hydrodynamics governed by the Navier–Stokes (NS) equations. Firstly, general comments on turbulence (as a statistical-mechanical problem) will be made. Then, I shall attempt to provide some hints (rather than definite answers) to a series of questions: What is generally the source of turbulence? What are the main statistical features of turbulence? How to deal with turbulence? Much more elaborated developments and references may be sought in the following books (among many others) dealing with turbulence:

- a reference book on the physics of turbulence: *A first course in turbulence* by H. Tennekes and J. L. Lumley, MIT Press, Cambridge, USA (1972)
- a reference book on turbulence as a statistical-mechanical problem: *Turbulence: The Legacy of A. N. Kolmogorov* by U. Frisch, Cambridge University Press, Cambridge, UK (1995)
- a reference textbook on the modelling of turbulence: *Turbulent flows* by S. Pope, Cambridge University Press, Cambridge, UK (2000)

Structure Formation in Astrophysics. ed. G. Chabrier. Published by Cambridge University Press.

- a reference book on the numerical modelling of turbulence: *Large-eddy simulation for incompressible flows – An introduction* by P. Sagaut, Springer-Verlag, Scientific Computation series (2005).

1.1 General comments on turbulence

Turbulence is employed to label flows with the common characteristics of complexity and disorder. *Complexity* refers to the complicated swirling motion of the fluid, the ability to distort material fluid elements into complex convoluted geometries. *Disorder* is related to this dynamics being random (or unpredictable). In this respect, turbulence should be approached from the viewpoint of statistical mechanics (Monin & Yaglom 1975).

Complexity in turbulence has to do with the essential non-linearity arising from the advection term in the dynamical equations:

$$\partial_t \mathbf{u}(\mathbf{x}, t) + (\mathbf{u}(\mathbf{x}, t) \cdot \nabla)\, \mathbf{u}(\mathbf{x}, t) = \textit{forces by unit mass.}$$

$\mathbf{u}(\mathbf{x}, t)$ denotes the velocity field. By recasting this term in the Fourier space (and assuming local homogeneity), one gets

$$\partial_t \widehat{u}_i(\mathbf{k}, t) = -ik_j \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \widehat{u}_i(\mathbf{p}, t)\widehat{u}_j(\mathbf{q}, t) + \ldots,$$

where the summation is over all allowed wavevectors $\mathbf{p}$ and $\mathbf{q}$. Thus, the time evolution of mode $\mathbf{k}$ is a priori driven by the triadic interactions with all modes such that $\mathbf{p} + \mathbf{q} = \mathbf{k}$. A specific feature of turbulence (as a dynamical system) lies in the impossibility to reduce this interaction to a restricted set of interacting modes $\mathbf{p}$ and $\mathbf{q}$. On the contrary, *strong interactions* with all triads of modes must be considered. Furthermore, long-range (phase) coherency between Fourier modes is expected to play an essential role; it is heuristically connected to the concentration of vorticity into intense thin fluid structures such as vortex filaments (Figure 1.1).

Disorder in turbulence has to do with a strong departure from absolute statistical equilibrium. From a theoretical viewpoint, the statistical problem of turbulence is a priori well posed if at initial time t_0 the mean velocity $U_i(\mathbf{x}, t_0)$ and the two-point correlation function

$$R_{ij}(\mathbf{x}, t_0; \mathbf{x}', t_0) \equiv \langle u_i(\mathbf{x}, t_0) u_j(\mathbf{x}', t_0)\rangle$$

are prescribed for all $\mathbf{x}$ and $\mathbf{x}'$ and if it is assumed that the (multivariate) distribution of the turbulent (fluctuating) velocity field $u_i(\mathbf{x}, t_0)$ is normal. However, at times $t > t_0$, it is observed that the multivariate distribution of $u_i(\mathbf{x}, t)$ strongly deviates from the normal distribution due to statistical correlations generated by

Fig. 1.1. Vortex filaments in a turbulent jet visualized by micro-bubbles. Courtesy of Olivier Cadot, École nationale supérieure de techniques avancées, Paris.

the hydrodynamical forces. Analytically, this departure from the normal distribution is governed by the entire infinite sequence of statistical equations (deduced from the NS equations) for all multi-point multi-time correlation functions

$$R_{ijk\ldots}(\mathbf{x}, t; \mathbf{x}', t'; \mathbf{x}'', t''; \ldots) \equiv \langle u_i(\mathbf{x}, t) u_j(\mathbf{x}', t') u_k(\mathbf{x}'', t'') \ldots \rangle .$$

An identified fundamental issue resides in finding an appropriate closure condition to convert this infinite hierarchy into a closed (low-dimensional) subset of equations and eventually solve it.

In practice, turbulence is often investigated from a phenomenological standpoint, i.e. starting from hypotheses motivated by experimental and numerical observations. This line of study has yielded very fruitful results during the past half century and continues to expand nowadays (in absence of any successful statistical theory).

1.2 What is the source of turbulence?

Turbulence is generally triggered by the inhomogeneities of the flow (by the mean velocity field not being uniform in space). Boundary conditions are often responsible for such inhomogeneities (Figure 1.2).

In order to dissect this mechanism, let us consider the flow over a (solid) flat plate. Because of the no-slip condition at the boundary, the velocity field necessarily decreases to zero in the vicinity of the plate. This implies a strong gradient of the mean velocity in the direction perpendicular to the plate. From kinematic considerations, this strong gradient may be viewed as a vorticity sheet attached to the plate. This sheet is generically unstable and generates streaky structures that

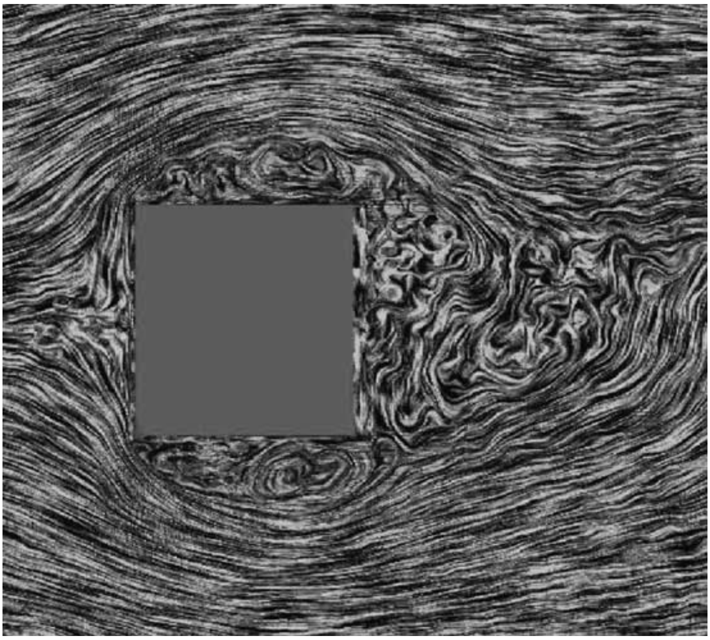

Fig. 1.2. Streamlines of a turbulent flow (from left to right) around an obstacle. It appears that turbulence originates in the vicinity of the obstacle.

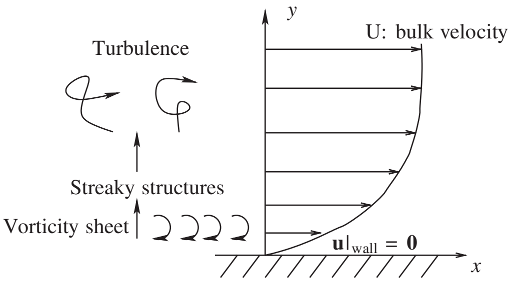

Fig. 1.3. A solid boundary may be viewed as a source of vorticity which is generically unstable, detaches from the boundary and sustains turbulence in the bulk.

eventually detach, interact and contribute to sustain turbulence in the bulk of the flow (Figure 1.3).

As rule of thumb, one may claim that a key ingredient involved in the generation of turbulence is the inhomogeneity of the mean flow, i.e. strong mean velocity gradients, and the instability of these gradients. This mechanism also tells us that it is important to learn about the mean flow before getting to the turbulent fluctuations.

1.3 What are the main statistical features of turbulence?

This section is devoted to the statistical features of turbulent motions, usually viewed as the net result of the interaction of a *gas of turbulent eddies*. By analogy with a molecule, an eddy may be seen as a glob of fluid of a given size, or (spatial) scale, that has a certain structure and life history of its own. This interaction is unpredictable in detail; however, statistically distinct properties can be identified and profitably examined. The need for a statistical description arises from both the *intrinsic* complexity of individual solutions and the instability of these solutions to infinitesimal perturbations (in the initial and boundary conditions). This makes it natural to examine ensemble of realizations rather than each individual realization and seek for robust statistical features insensitive to the details of perturbations.

It is characteristic of turbulence that turbulent eddies are distributed over a wide range of size scales and associated turn-over timescales. This range spreads from the *integral scale* L_S, which refers to the size of the largest eddies in the flow (triggered by the inhomogeneities of the mean flow as seen previously), to the *elementary scale* η, which nails down the size of the smallest eddies. The macroscale L_S can be estimated by equalling the timescale related to the local mean velocity gradient, $1/\|\nabla \mathbf{U}\|$, and the turn-over time of the largest eddies, L_S/u, where u denotes the root-mean-squared turbulent velocity. Let us note that the norm of the mean velocity gradient is often referred to as the *shear* in the literature, and L_S is called the *shear length scale*. The elementary microscale η is the viscous cut-off scale. Formally, it is estimated by equalling the viscous timescale η^2/ν, where ν is the kinematic viscosity of the fluid, and the turn-over time of the smallest eddies. This latter must be modelled (see p. 9).

The range of excited scales

$$L_S \sim \frac{u}{\|\nabla \mathbf{U}\|} \geq r \geq \eta$$

is called the *inertial range*. In that range, interactions between turbulent eddies result in an effective transfer of kinetic energy from the large scales (comparable to L_S), where energy is fed into turbulence, to the small scales (comparable to η), where this energy is dissipated by molecular viscosity. In 1941, Kolmogorov envisaged to describe this mechanism by a self-similar cascade of kinetic energy, which is local in scale and in which all statistical information concerning the large scales is lost (except for the mean energy-cascade rate ε). Kolmogorov's theory yields the celebrated *universal law* for the kinetic energy spectrum (in wavenumber k):

$$E(k) = C\varepsilon^{2/3}k^{-5/3},$$

where C is a universal constant (Kolmogorov 1941). It is worth noting that this law does not include any characteristic scale; this refers to the idea that the energy cascade is a self-similar process in scale.

The Kolmogorov's energy spectrum represents a form for the inertial range of wavenumbers, in which energy is transferred from small to large wavenumbers by the process of vortex stretching. Indeed, the chaotic nature of turbulence tends to separate any two fluid elements initially near to each other. Consequently, there is a tendency to stretch initial vorticity into ever-elongated and thinning structures, until viscosity stops the thinning (Figure 1.4). By Kelvin's theorem, if the cross section of a vortex structure decreases under stretching, the fluid in the vortex must spin faster. The combination of stretching and spin-up means a transfer of energy from lower to higher wavenumbers. The viscous cut-off scale η, which identifies the bottom scale of the energy cascade, is given (from dimensional arguments) by

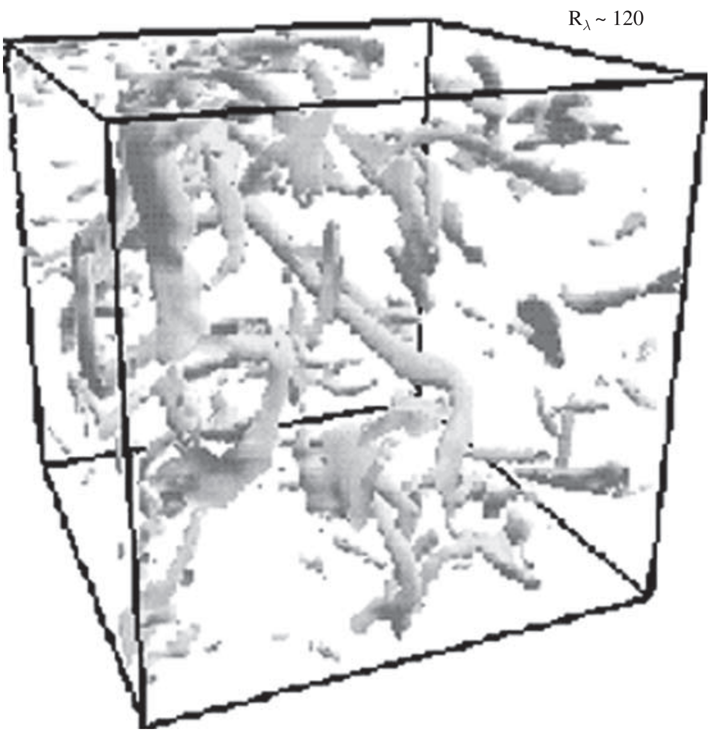

Fig. 1.4. Snapshot of high-enstrophy isosurfaces from a numerical simulation of three-dimensional turbulence; the local enstrophy is defined by $|\vec{\nabla} \times \vec{u}(\vec{x}, t)|^2$. The swirling activity of the flow concentrates into very localized elongated and thin fluid structures: the *vortex filaments* (E. Lévêque).

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}.$$

A large body of experimental and numerical measurements corroborate the Kolmogorov's energy spectrum (Figure 1.5). However, higher-order statistical correlations are not universal in the sense of Kolmogorov's hypothesis. These discrepancies are rooted in the fact that the cascade of energy is actually a highly non-uniform process in space and time (Figure 1.6). This feature is usually referred to as *intermittency* in the literature. From the viewpoint of statistical mechanics, intermittency implies that the macroscopic parameter ε is not sufficient to characterize the energy-cascade state of turbulence, but fluctuations of $\varepsilon(\mathbf{x}, t)$ should be taken into account. Once Kolmogorov's *mean field theory* is abandoned, a Pandora's box of possibilities is opened and a specific contact with the dynamics of turbulence (solution of the NS equations) must be achieved. Current models have not succeeded to establish this contact. They essentially rely on plausible hypotheses but fail to relate themselves to the actual dynamics. More recent works attempt to correlate turbulent high-order velocity correlations with the presence of highly

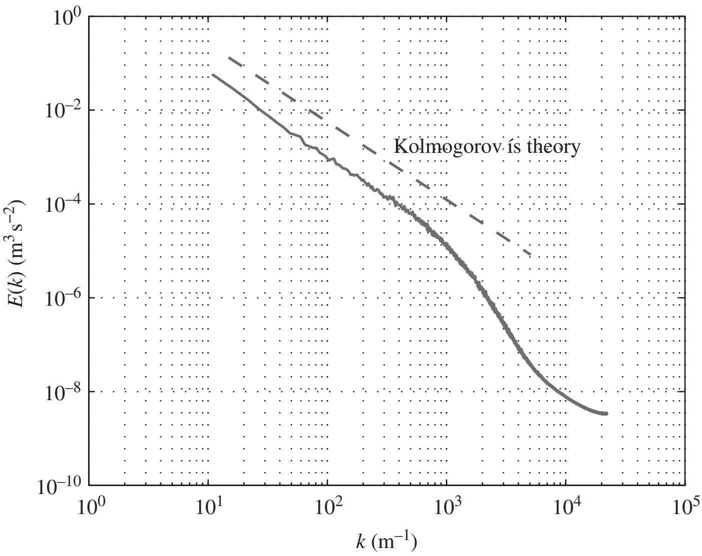

Fig. 1.5. The density (in wavenumber k) of turbulent kinetic energy $E(k)$ exhibits a universal $k^{-5/3}$ decrease (obtained from a laboratory experiment, courtesy of C. Baudet and S. Ciliberto, ENS-Lyon, France) in agreement with the Kolmogorov's theory. Turbulent motions are strongly damped by molecular viscosity at very large wavenumbers. The energy is not equally distributed among Fourier modes: $E(k) \sim k^2$ would be expected at statistical thermodynamic equilibrium. Turbulence is a far-from-equilibrium system.

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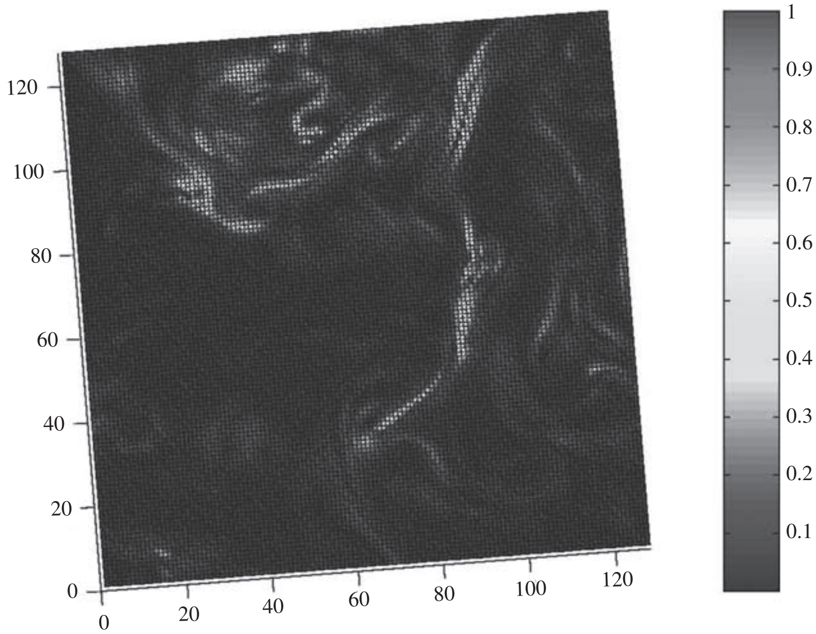

Fig. 1.6. Slice of a snapshot of the energy-dissipation rate $\varepsilon(\mathbf{x}, t)$ obtained from a numerical simulation of the NS equations (E. Lévêque). The magnitude of $\varepsilon(\mathbf{x}, t)$ is represented by a grayscale bar ranging from 0 to 1. Energy dissipation is concentrated on fine structures; it is not uniformly distributed.

coherent dynamical structures, the so-called *She–Lévêque model* (She & Lévêque 1994), for instance.

High-order multi-point correlations of turbulent velocity fluctuations are commonly investigated through the velocity structure functions, defined by

$$S_p(r) \equiv \left\langle |\mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x} + \mathbf{r}, t)|^p \right\rangle$$

for $p = 1, 2, \ldots$ and the separation scale r within the inertial range. It is observed both experimentally and numerically that the $S_p(r)$'s exhibit power-law scalings:

$$S_p(r) \sim r^{\zeta_p}$$

and the scaling exponents ζ_p are found to be universal. The She–Lévêque model relates the set of scaling exponents ζ_p to the presence of vortex filaments and yields a formula without any adjustable parameter:

$$\zeta_p = \frac{p}{9} + 2\left[1 - \left(\frac{2}{3}\right)^{p/3}\right].$$

This model, which is found in very good agreement with experimental and numerical data, establishes a concrete link between the dynamics and the statistics of turbulence. Furthermore, its formulation is very general and can apply to a vast class of turbulent systems. In astrophysics, it has been successfully employed to relate the statistics to coherent dynamical structures in the interstellar medium, in the solar wind or in cosmic rays.

1.4 How to deal with turbulence?

1.4.1 The Reynolds-averaged Navier–Stokes equations

How to account for turbulence in the mean flow? This fundamental question was raised by Osborne Reynolds more than a century ago (Reynolds 1895) and remains mostly open today.

In the turbulent regime, the mean flow is solution of the NS equations complemented by a force which encompasses the exchange of momentum between the mean flow and the turbulent agitation:

$$\rho \frac{\mathrm{d}U_i}{\mathrm{d}t} = \mathrm{NS}(\mathbf{U}) - \rho \frac{\partial \overline{u_i u_j}}{\partial x_j}.$$

$-\rho\overline{u_i u_j}$ is termed the *Reynolds stress* (the overbar represents the statistical mean value). These equations are commonly called the *Reynolds-averaged Navier–Stokes* (RANS) equations. It is necessary to express the Reynolds stress in terms of the mean velocity field in order to close the RANS equations.

1.4.1.1 The concept of turbulent viscosity

The idea behind the introduction of a turbulent viscosity is to treat the deviatoric (traceless) part of the Reynolds stress like the viscous stress in a Newtonian fluid:

$$-\rho\overline{u_i u_j} + \frac{1}{3}\rho\overline{u_k u_k}\,\delta_{ij} = 2\rho\nu_{\mathrm{turb}}\overline{S}_{ij},$$

where ν_{turb} is the (kinematic) turbulent viscosity and $\overline{S}_{ij}$ is the mean rate-of-strain tensor (the symmetric part of the velocity gradient tensor). The turbulent viscosity $\nu_{\mathrm{turb}}(\mathbf{x}, t)$ depends on the position $\mathbf{x}$ in the flow and time t; it is a property of the flow not of the fluid. The introduction of a turbulent viscosity relies on the hypothesis of an internal friction (related to the turbulent agitation of the fluid) responsible for the diffusive transport of momentum from the rapid to the slow mean-flow regions.

Dimensionally, ν_{turb} is equivalent to the product of a velocity and a length. This suggests that (by analogy to the kinetic theory for gases)

$$\nu_{\mathrm{turb}} = u\,\ell,$$

where u and ℓ would represent the characteristic velocity and length of the turbulent agitation. In the *mixing-length* model, u and ℓ are specified on the basis of the