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HILBERT TRANSFORMS

The Hilbert transform arises widely in a variety of applications, including problems in aerodynamics, condensed matter physics, optics, fluids, and engineering.

This work, written in an easy-to-use style, is destined to become the definitive reference on the subject. It contains a thorough discussion of all the common Hilbert transforms, mathematical techniques for evaluating them, and a detailed discussion of their application. Especially valuable features are the tabulation of analytically evaluated Hilbert transforms, and an atlas that immediately illustrates how the Hilbert transform alters a function. These will provide useful and convenient resources for researchers.

A collection of exercises is provided for the reader to test comprehension of the material in each chapter. The bibliography is an extensive collection of references to both the classical mathematical papers, and to a diverse array of applications.

FREDERICK W. KING is a Professor in the Department of Chemistry at the University of Wisconsin-Eau Claire.

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Hilbert transforms

Volume 1

FREDERICK W. KING

University of Wisconsin-Eau Claire



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To the memory of my mother

Contents

<i>Preface</i>	<i>page</i> xxi
<i>List of symbols</i>	xxv
<i>List of abbreviations</i>	xxxviii

Volume I

1 Introduction	1
1.1 Some common integral transforms	1
1.2 Definition of the Hilbert transform	1
1.3 The Hilbert transform as an operator	4
1.4 Diversity of applications of the Hilbert transform	6
Notes	8
Exercises	9
2 Review of some background mathematics	11
2.1 Introduction	11
2.2 Order symbols $O()$ and $o()$	11
2.3 Lipschitz and Hölder conditions	12
2.4 Cauchy principal value	13
2.5 Fourier series	14
2.5.1 Periodic property	14
2.5.2 Piecewise continuous functions	15
2.5.3 Definition of Fourier series	16
2.5.4 Bessel's inequality	19
2.6 Fourier transforms	19
2.6.1 Definition of the Fourier transform	19
2.6.2 Convolution theorem	21
2.6.3 The Parseval and Plancherel formulas	21
2.7 The Fourier integral	22

2.8	Some basic results from complex variable theory	23
2.8.1	Integration of analytic functions	27
2.8.2	Cauchy integral theorem	29
2.8.3	Cauchy integral formula	30
2.8.4	Jordan's lemma	30
2.8.5	The Laurent expansion	31
2.8.6	The Cauchy residue theorem	33
2.8.7	Entire functions	34
2.9	Conformal mapping	37
2.10	Some functional analysis basics	39
2.10.1	Hilbert space	42
2.10.2	The Hardy space H^p	43
2.10.3	Topological space	44
2.10.4	Compact operators	45
2.11	Lebesgue measure and integration	45
2.11.1	The notion of measure	48
2.12	Theorems due to Fubini and Tonelli	55
2.13	The Hardy–Poincaré–Bertrand formula	57
2.14	Riemann–Lebesgue lemma	61
2.15	Some elements of the theory of distributions	63
2.15.1	Generalized functions as sequences of functions	65
2.15.2	Schwartz distributions	68
2.16	Summation of series: convergence accelerator techniques	70
2.16.1	Richardson extrapolation	71
2.16.2	The Levin sequence transformations	74
	Notes	77
	Exercises	80
3	Derivation of the Hilbert transform relations	83
3.1	Hilbert transforms – basic forms	83
3.2	The Poisson integral for the half plane	85
3.3	The Poisson integral for the disc	89
3.3.1	The Poisson kernel for the disc	91
3.4	Hilbert transform on the real line	94
3.4.1	Conditions on the function f	96
3.4.2	The Phragmén–Lindelöf theorem	100
3.4.3	Some examples	101
3.5	Transformation to other limits	104
3.6	Cauchy integrals	107
3.7	The Plemelj formulas	111
3.8	Inversion formula for a Cauchy integral	112
3.9	Hilbert transform on the circle	114

<i>Contents</i>		ix
3.10	Alternative approach to the Hilbert transform on the circle	115
3.11	Hardy's approach	118
3.11.1	Hilbert transform on \mathbb{R}	120
3.12	Fourier integral approach to the Hilbert transform on \mathbb{R}	122
3.13	Fourier series approach	129
3.14	The Hilbert transform for periodic functions	132
3.15	Cancellation behavior for the Hilbert transform	135
	Notes	141
	Exercises	142
4	Some basic properties of the Hilbert transform	145
4.1	Introduction	145
4.1.1	Complex conjugation property	145
4.1.2	Linearity	145
4.2	Hilbert transforms of even or odd functions	146
4.3	Skew-symmetric character of Hilbert transform pairs	147
4.4	Inversion property	148
4.5	Scale changes	150
4.5.1	Linear scale changes	150
4.5.2	Some nonlinear scale transformations for the Hilbert transform	150
4.6	Translation, dilation, and reflection operators	155
4.7	The Hilbert transform of the product $x^n f(x)$	160
4.8	The Hilbert transform of derivatives	164
4.9	Convolution property	167
4.10	Titchmarsh formulas of the Parseval type	170
4.11	Unitary property of H	174
4.12	Orthogonality property	174
4.13	Hilbert transforms via series expansion	177
4.14	The Hilbert transform of a product of functions	181
4.15	The Hilbert transform product theorem (Bedrosian's theorem)	184
4.16	A theorem due to Tricomi	187
4.17	Eigenvalues and eigenfunctions of the Hilbert transform operator	195
4.18	Projection operators	199
4.19	A theorem due to Akhiezer	200
4.20	The Riesz inequality	203
4.21	The Hilbert transform of functions in L^1 and in L^∞	211
4.21.1	The L^∞ case	214
4.22	Connection between Hilbert transforms and causal functions	215
4.23	The Hardy–Poincaré–Bertrand formula revisited	223
4.24	A theorem due to McLean and Elliott	226

4.25	The Hilbert–Stieltjes transform	231
4.26	A theorem due to Stein and Weiss	241
	Notes	246
	Exercises	248
5	Relationship between the Hilbert transform and some common transforms	252
5.1	Introduction	252
5.2	Fourier transform of the Hilbert transform	252
5.3	Even and odd Hilbert transform operators	258
5.4	The commutator $[\mathcal{F}, H]$	261
5.5	Hartley transform of the Hilbert transform	262
5.6	Relationship between the Hilbert transform and the Stieltjes transform	265
5.7	Relationship between the Laplace transform and the Hilbert transform	267
5.8	Mellin transform of the Hilbert transform	269
5.9	The Fourier allied integral	275
5.10	The Radon transform	277
	Notes	285
	Exercises	286
6	The Hilbert transform of periodic functions	288
6.1	Introduction	288
6.2	Approach using infinite product expansions	289
6.3	Fourier series approach	291
6.4	An operator approach to the Hilbert transform on the circle	292
6.5	Hilbert transforms of some standard kernels	296
6.6	The inversion formula	301
6.7	Even and odd periodic functions	303
6.8	Scale changes	304
6.9	Parseval-type formulas	305
6.10	Convolution property	307
6.11	Connection with Fourier transforms	309
6.12	Orthogonality property	310
6.13	Eigenvalues and eigenfunctions of the Hilbert transform operator	311
6.14	Projection operators	312
6.15	The Hardy–Poincaré–Bertrand formula	313
6.16	A theorem due to Privalov	316
6.17	The Marcel Riesz inequality	318
6.18	The partial sum of a Fourier series	323
6.19	Lusin’s conjecture	325
	Notes	328
	Exercises	329

<i>Contents</i>		xi
7	Inequalities for the Hilbert transform	331
7.1	The Marcel Riesz inequality revisited	331
7.1.1	Hilbert's integral	335
7.2	A Kolmogorov inequality	335
7.3	A Zygmund inequality	340
7.4	A Bernstein inequality	343
7.5	The Hilbert transform of a function having a bounded integral and derivative	350
7.6	Connections between the Hilbert transform on \mathbb{R} and \mathbb{T}	352
7.7	Weighted norm inequalities for the Hilbert transform	354
7.8	Weak-type inequalities	364
7.9	The Hardy–Littlewood maximal function	368
7.10	The maximal Hilbert transform function	373
7.11	A theorem due to Helson and Szegö	386
7.12	The A_p condition	388
7.13	A theorem due to Hunt, Muckenhoupt, and Wheeden	395
7.13.1	Weighted norm inequalities for H_e and H_o	399
7.14	Weighted norm inequalities for the Hilbert transform of functions with vanishing moments	402
7.15	Weighted norm inequalities for the Hilbert transform with two weights	403
7.16	Some miscellaneous inequalities for the Hilbert transform	408
	Notes	413
	Exercises	416
8	Asymptotic behavior of the Hilbert transform	419
8.1	Asymptotic expansions	419
8.2	Asymptotic expansion of the Stieltjes transform	420
8.3	Asymptotic expansion of the one-sided Hilbert transform	422
	Notes	434
	Exercises	434
9	Hilbert transforms of some special functions	438
9.1	Hilbert transforms of special functions	438
9.2	Hilbert transforms involving Legendre polynomials	438
9.3	Hilbert transforms of the Hermite polynomials with a Gaussian weight	446
9.4	Hilbert transforms of the Laguerre polynomials with a weight function $H(x)e^{-x}$	449
9.5	Other orthogonal polynomials	452
9.6	Bessel functions of the first kind	453
9.7	Bessel functions of the first and second kind for non-integer index	459

9.8	The Struve function	460
9.9	Spherical Bessel functions	462
9.10	Modified Bessel functions of the first and second kind	465
9.11	The cosine and sine integral functions	470
9.12	The Weber and Anger functions	470
	Notes	471
	Exercises	472
10	Hilbert transforms involving distributions	474
10.1	Some basic distributions	474
10.2	Some important spaces for distributions	478
10.3	Some key distributions	483
10.4	The Fourier transform of some key distributions	487
10.5	A Parseval-type formula approach to HT	490
10.6	Convolution operation for distributions	491
10.7	Convolution and the Hilbert transform	495
10.8	Analytic representation of distributions	498
10.9	The inversion formula	502
10.10	The derivative property	504
10.11	The Fourier transform connection	505
10.12	Periodic distributions: some preliminary notions	508
10.13	The Hilbert transform of periodic distributions	515
10.14	The Hilbert transform of ultradistributions and related ideas	516
	Notes	518
	Exercises	519
11	The finite Hilbert transform	521
11.1	Introduction	521
11.2	Alternative formulas: the cosine form	523
11.2.1	A result due to Hardy	527
11.3	The cotangent form	527
11.4	The inversion formula: Tricomi's approach	529
11.4.1	Inversion of the finite Hilbert transform for the interval $(0, 1)$	536
11.5	Inversion by a Fourier series approach	541
11.6	The Riemann problem	543
11.7	The Hilbert problem	544
11.8	The Riemann–Hilbert problem	544
11.8.1	The index of a function	546
11.9	Carleman's approach	550

<i>Contents</i>		xiii
11.10	Some basic properties of the finite Hilbert transform	552
11.10.1	Even–odd character	552
11.10.2	Inversion property	552
11.10.3	Scale changes	553
11.10.4	Finite Hilbert transform of the product $x^n f(x)$	554
11.10.5	Derivative of the finite Hilbert transform	555
11.10.6	Convolution property	556
11.10.7	Fourier transform of the finite Hilbert transform	557
11.10.8	Parseval-type identities	559
11.10.9	Orthogonality property	562
11.10.10	Eigenfunctions and eigenvalues of the finite Hilbert transform operator	563
11.11	Finite Hilbert transform of the Legendre polynomials	563
11.12	Finite Hilbert transform of the Chebyshev polynomials	567
11.13	Contour integration approach to the derivation of some finite Hilbert transforms	570
11.14	The thin airfoil problem	577
11.15	The generalized airfoil problem	582
11.16	The cofinite Hilbert transform	582
	Notes	584
	Exercises	585
12	Some singular integral equations	588
12.1	Introduction	588
12.2	Fredholm equations of the first kind	589
12.3	Fredholm equations of the second kind	592
12.4	Fredholm equations of the third kind	594
12.5	Fourier transform approach to solving singular integral equations	599
12.6	A finite Hilbert transform integral equation	601
12.7	The one-sided Hilbert transform	609
12.7.1	Eigenfunctions and eigenvalues of the one-sided Hilbert transform operator	612
12.8	Fourier transform approach to the inversion of the one-sided Hilbert transform	612
12.9	An inhomogeneous singular integral equation for H_1	614
12.10	A nonlinear singular integral equation	617
12.11	The Peierls–Nabarro equation	618
12.12	The sine–Hilbert equation	620
12.13	The Benjamin–Ono equation	624
12.13.1	Conservation laws	627
12.14	Singular integral equations involving distributions	630
	Notes	632
	Exercises	633

13 Discrete Hilbert transforms	637
13.1 Introduction	637
13.2 The discrete Fourier transform	637
13.3 Some properties of the discrete Fourier transform	640
13.4 Evaluation of the DFT	641
13.5 Relationship between the DFT and the Fourier transform	643
13.6 The Z transform	644
13.7 Z transform of a product	647
13.8 The Hilbert transform of a discrete time signal	649
13.9 Z transform of a causal sequence	652
13.10 Fourier transform of a causal sequence	656
13.11 The discrete Hilbert transform in analysis	660
13.12 Hilbert's inequality	661
13.13 Alternative approach to the discrete Hilbert transform	666
13.14 Discrete analytic functions	675
13.15 Weighted discrete Hilbert transform inequalities	679
Notes	680
Exercises	681
14 Numerical evaluation of Hilbert transforms	684
14.1 Introduction	684
14.2 Some elementary transformations for Cauchy principal value integrals	684
14.3 Some classical formulas for numerical quadrature	688
14.3.1 A Maclaurin-type formula	688
14.3.2 The trapezoidal rule	690
14.3.3 Simpson's rule	691
14.4 Gaussian quadrature: some basics	691
14.5 Gaussian quadrature: implementation procedures	694
14.6 Specialized Gaussian quadrature: application to the Hilbert transform	701
14.6.1 Error estimates	706
14.7 Specialized Gaussian quadrature: application to H_e and H_o	708
14.8 Numerical integration of the Fourier transform	709
14.9 The fast Fourier transform: numerical implementation	711
14.10 Hilbert transform via the fast Fourier transform	712
14.11 The Hilbert transform via the allied Fourier integral	712
14.12 The Hilbert transform via conjugate Fourier series	713
14.13 The Hilbert transform of oscillatory functions	717
14.14 An eigenfunction expansion	723
14.15 The finite Hilbert transform	727

Contents xv

Notes	730
Exercises	732
Appendix 14.1	735
Appendix 14.2	744
<i>References</i>	745
<i>Author index</i>	824
<i>Subject index</i>	840

Volume II

15 Hilbert transforms in E^n	1
15.1 Definition of the Hilbert transform in E^n	1
15.2 Definition of the n -dimensional Hilbert transform	5
15.3 The double Hilbert transform	8
15.4 Inversion property for the n -dimensional Hilbert transform	10
15.5 Derivative of the n -dimensional Hilbert transform	11
15.6 Fourier transform of the n -dimensional Hilbert transform	12
15.7 Relationship between the n -dimensional Hilbert transform and translation and dilation operators	14
15.8 The Parseval-type formula	16
15.9 Eigenvalues and eigenfunctions of the n -dimensional Hilbert transform	17
15.10 Periodic functions	18
15.11 A Calderón–Zygmund inequality	21
15.12 The Riesz transform	25
15.13 The n -dimensional Hilbert transform of distributions	32
15.14 Connection with analytic functions	38
Notes	41
Exercises	42
16 Some further extensions of the classical Hilbert transform	44
16.1 Introduction	44
16.2 An extension due to Redheffer	44
16.3 Kober's definition for the L^∞ case	47
16.4 The Boas transform	49
16.4.1 Connection with the Hilbert transform	49
16.4.2 Parseval-type formula for the Boas transform	51
16.4.3 Iteration formula for the Boas transform	52
16.4.4 Riesz-type bound for the Boas transform	52
16.4.5 Fourier transform of the Boas transform	53
16.4.6 Two theorems due to Boas	54
16.4.7 Inversion of the Boas transform	55
16.4.8 Generalization of the Boas transform	56

16.5	The bilinear Hilbert transform	58
16.6	The vectorial Hilbert transform	60
16.7	The directional Hilbert transform	60
16.8	Hilbert transforms along curves	62
16.9	The ergodic Hilbert transform	63
16.10	The helical Hilbert transform	66
16.11	Some miscellaneous extensions of the Hilbert transform	67
	Notes	69
	Exercises	70
17	Linear systems and causality	73
17.1	Systems	73
17.2	Linear systems	73
17.3	Sequential systems	79
17.4	Stationary systems	79
17.5	Primitive statement of causality	80
17.6	The frequency domain	81
17.7	Connection to analyticity	83
	17.7.1 A generalized response function	87
17.8	Application of a theorem due to Titchmarsh	90
17.9	An acausal example	93
17.10	The Paley–Wiener log-integral theorem	95
17.11	Extensions of the causality concept	102
17.12	Basic quantum scattering: causality conditions	105
17.13	Extension of Titchmarsh’s theorem for distributions	110
	Notes	116
	Exercises	117
18	The Hilbert transform of waveforms and signal processing	119
18.1	Introductory ideas on signal processing	119
18.2	The Hilbert filter	121
18.3	The auto-convolution, cross-correlation, and auto-correlation functions	123
18.4	The analytic signal	126
18.5	Amplitude modulation	135
18.6	The frequency domain	138
18.7	Some useful step and pulse functions	139
	18.7.1 The Heaviside function	139
	18.7.2 The signum function	142
	18.7.3 The rectangular pulse function	143
	18.7.4 The triangular pulse function	145
	18.7.5 The sinc pulse function	145

<i>Contents</i>		xvii
18.8	The Hilbert transform of step functions and pulse forms	146
18.9	The fractional Hilbert transform: the Lohmann–Mendlovic–Zalevsky definition	147
18.10	The fractional Fourier transform	149
18.11	The fractional Hilbert transform: Zayed’s definition	159
18.12	The fractional Hilbert transform: the Cusmariu definition	160
18.13	The discrete fractional Fourier transform	163
18.14	The discrete fractional Hilbert transform	168
18.15	The fractional analytic signal	169
18.16	Empirical mode decomposition: the Hilbert–Huang transform	170
	Notes	178
	Exercises	180
19	Kramers–Kronig relations	182
19.1	Some background from classical electrodynamics	182
19.2	Kramers–Kronig relations: a simple derivation	184
19.3	Kramers–Kronig relations: a more rigorous derivation	190
19.4	An alternative approach to the Kramers–Kronig relations	197
19.5	Direct derivation of the Kramers–Kronig relations on the interval $[0, \infty)$	199
19.6	The refractive index: Kramers–Kronig relations	201
19.7	Application of Herglotz functions	208
19.8	Conducting materials	216
19.9	Asymptotic behavior of the dispersion relations	219
19.10	Sum rules for the dielectric constant	222
19.11	Sum rules for the refractive index	227
19.12	Application of some properties of the Hilbert transform	231
19.13	Sum rules involving weight functions	236
19.14	Summary of sum rules for the dielectric constant and refractive index	239
19.15	Light scattering: the forward scattering amplitude	239
	Notes	247
	Exercises	250
20	Dispersion relations for some linear optical properties	252
20.1	Introduction	252
20.2	Dispersion relations for the normal-incident reflectance and phase	252
20.3	Sum rules for the reflectance and phase	263
20.4	The conductance: dispersion relations	267
20.5	The energy loss function: dispersion relations	269
20.6	The permeability: dispersion relations	271

xviii	<i>Contents</i>	
20.7	The surface impedance: dispersion relations	274
20.8	Anisotropic media	278
20.9	Spatial dispersion	280
20.10	Fourier series representation	290
20.11	Fourier series approach to the reflectance	294
20.12	Fourier and allied integral representation	298
20.13	Integral inequalities	300
	Notes	303
	Exercises	304
21	Dispersion relations for magneto-optical and natural optical activity	306
21.1	Introduction	306
21.2	Circular polarization	307
21.3	The complex refractive indices N_+ and N_-	309
21.4	Are there dispersion relations for the individual complex refractive indices N_+ and N_- ?	316
21.5	Magnetic optical activity: Faraday effect and magnetic circular dichroism	319
21.6	Sum rules for magneto-optical activity	323
21.7	Magnetorefectivity	325
21.8	Optical activity	330
21.9	Dispersion relations for optical activity	345
21.10	Sum rules for optical activity	346
	Notes	348
	Exercises	349
22	Dispersion relations for nonlinear optical properties	351
22.1	Introduction	351
22.2	Some types of nonlinear optical response	357
22.3	Classical description: the anharmonic oscillator	359
22.4	Density matrix treatment	362
22.5	Asymptotic behavior for the nonlinear susceptibility	372
22.6	One-variable dispersion relations for the nonlinear susceptibility	377
22.7	Experimental verification of the dispersion relations for the nonlinear susceptibility	384
22.8	Dispersion relations in two variables	386
22.9	n -dimensional dispersion relations	387
22.10	Situations where the dispersion relations do not hold	388
22.11	Sum rules for the nonlinear susceptibilities	392
22.12	Summary of sum rules for the nonlinear susceptibilities	395
22.13	The nonlinear refractive index and the nonlinear permittivity	395

<i>Contents</i>		xix
Notes		403
Exercises		404
23 Some further applications of Hilbert transforms		406
23.1 Introduction		406
23.2 Hilbert transform spectroscopy		406
23.2.1 The Josephson junction		406
23.2.2 Absorption enhancement		410
23.3 The phase retrieval problem		411
23.4 X-ray crystallography		417
23.5 Electron–atom scattering		422
23.5.1 Potential scattering		422
23.5.2 Dispersion relations for potential scattering		425
23.5.3 Dispersion relations for electron–H atom scattering		428
23.6 Magnetic resonance applications		433
23.7 DISPA analysis		435
23.8 Electrical circuit analysis		437
23.9 Applications in acoustics		444
23.10 Viscoelastic behavior		447
23.11 Epilog		448
Notes		449
Exercises		451
Appendix 1 Table of selected Hilbert transforms		453
Appendix 2 Atlas of selected Hilbert transform pairs		534
<i>References</i>		547
<i>Author index</i>		626
<i>Subject index</i>		642

Preface

My objective in this book is to present an elementary introduction to the theory of the Hilbert transform and a selection of applications where this transform is applied. The treatment is directed primarily at mathematically well prepared upper division undergraduates in physics and related sciences, as well as engineering, and first-year graduate students in these areas. Undergraduate students with a major in applied mathematics will find material of interest in this work.

I have attempted to make the treatment self-contained. To that end, I have collected a number of topics for review in Chapter 2. A reader with a good undergraduate mathematics background could possibly skip over much of this chapter. For others, it might serve as a highly condensed review of material used later in the text. The principal background mathematics assumed of the reader is a solid foundation in basic calculus, including introductory differential equations, a course in linear and abstract algebra, some exposure to operator theory basics, and an introductory knowledge of complex variables. Readers with a few deficiencies in these areas will find a number of recommendations for further reading at the end of Chapter 2. Some of the applications discussed require the reader to be familiar with basic electrodynamics.

A focus of the book is on problem solving rather than on proving theorems. Theorems are, for the most part, not stated or proved in the most general form possible. The end-notes will typically provide additional reference sources of more detailed discussions about the various theorems presented. I have not attempted to sketch the proof of every theorem stated, but for the key results connected to the Hilbert transform, at minimum an outline of the essential elements is usually presented. Consistent with the problem-solving emphasis is that all the different techniques that I know for evaluating Hilbert transforms are displayed in the book.

I take the opportunity to introduce special functions in a number of settings. I do this for two reasons. Special functions occur widely in problems of great importance in many areas of physics and engineering, and, accordingly, it is essential that students gain exposure to this important area of mathematics. Since many Hilbert transforms evaluate to special functions, it is imperative that the reader know when to stop doing algebraic manipulations. I have incorporated several mathematical topics for which few or no applications are known to the writer. The selection process was governed

in part by the potential that I thought a particular area might have in problem solving, and I have done this with the full knowledge that crystal-ball reading is an art rather than a science!

The exercises are intended as a means for the reader to test his/her comprehension of the material in each chapter. The vast majority of the problems are by design routine applications of ideas discussed in the text. A small percentage of the problems are likely to be fairly challenging for an undergraduate reader, and a few problems could be labeled rather difficult. Most readers will have no trouble deciding when they have encountered an example of this latter group.

I have compiled an extensive table of Hilbert transforms of common mathematical functions. I hope this table will be useful in three ways. First, it serves as the answer key for a number of exercises that are placed throughout the text. Since many additional Hilbert transform pairs can be established by differentiation, or by appropriate multiplicative operations, etc., this table can be used to generate a great number of exercises, much to the delight of the reader. Second, I hope it will provide a useful reference source for those looking for the Hilbert transform of a particular function. Finally, for those searching for a particular Hilbert transform not present in the table, finding related transforms may give an idea on how to approach the evaluation, and give some clues as to whether a closed form expression in terms of standard functions is likely to be possible. In several sections the table includes a few specific cases followed by the general formula. This has been done to allow the reader to access the Hilbert transform of some of the simpler special cases as quickly as possible, rather than reducing a more complicated general formula.

The mini atlas of functions and the associated Hilbert transforms given in Appendix 2 is intended to provide a visual representation for a selection of Hilbert transform pairs. I hope this will be valuable for students in the applied sciences and engineering.

The reference list is rather extensive, but is not intended to be exhaustive. There are far too many published articles on Hilbert transforms to provide a complete set of references. I have attempted to give a generous number of references to applications. Many citations are given to the classical mathematical papers on the topics of the book, and for the serious student these works can be read with great profit. The Notes section at the end of each chapter gives a guide as to where to start reading for further information on topics discussed in the chapter. Elaborations and further details on the proofs of different theorems will often be located in the references cited in the end-notes.

My final task is to thank those who have helped. Logan Ausman, Dr. Matt Feldmann, Geir Helleloid, Dr. Kai-Erik Peiponen, Dr. Ignacio Porras, Dr. Jarkko Saarinen, and Corey Schuster read various chapters and made a number of useful suggestions to improve the presentation. Dr. Walter Reid and Dr. Jim Walker gave me some helpful comments on a preliminary draft of the first three chapters. Julia Boryskina and Hristina Ninova assisted with the translation of a number of technical papers. Several other students did translations and I offer a collective thanks to them.

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Preface

xxiii

Julia also provided assistance in the construction of the atlas of Hilbert transforms and with a number of the figures. Ali Elgindi did some numerical checking on the table of Hilbert transforms and Julia also did a few preliminary tests. Thanks are extended to Irene Pizzie for her efforts to improve the presentation.

The author would greatly appreciate if readers would bring to his attention any errors that escaped detection. The URL <http://www.chem.uwec.edu/king/> is the web address where corrections will be posted. It is the author's intention to maintain this site actively.

Symbols

The first occurrence or a definition is indicated by a section reference or an equation number. HT is an abbreviation for the tables of Hilbert transforms given in the Appendixes (Table 1.1).

$ a $	sum of the components of the multi-index a ; §15.5
$\arg z$	argument of a complex number; Eq. (2.69)
A_p	the A_p condition for $1 \leq p \leq \infty$; Eq. (7.377), §7.12
$b\Omega$	boundary of a bounded domain Ω ; §3.1
B	Boas transform operator; §16.4
B_n	generalization of the Boas transform operator; Eq. (16.84)
$\mathbf{B}(t)$	magnetic induction; §17.9
\mathcal{B}	generalization of the Boas transform operator; Eq. (16.80)
$B(a, b)$	beta function (Euler's integral of the first kind); Eq. (5.112), HT-01
$BV([a, b])$	class of functions that have bounded variation on the interval $[a, b]$; §4.25
C	designation for a contour (usually closed); §2.8.1.
C	a positive (often unspecified) constant; in derivations such a constant need not be the same at each occurrence, even though the same symbol is employed.
C	SI unit for charge, the coulomb, §19.1
\mathbb{C}	the set of complex numbers; §2.10
C_n	symmetry operation such that rotation by $2\pi/n$ leaves the system invariant; §21.3
C^∞	infinitely differentiable function for all points of \mathbb{R} ; §2.15.2
C_0^∞	class of functions that are infinitely differentiable with compact support; §2.15.2
C^k	class of functions that are continuously differentiable up to order k ; §2.15.2

xxvi

Symbols

C_0^k	class of functions that are continuously differentiable up to order k and have compact support; §2.15.2.
C_p	positive constant depending on the parameter p ; often not the same at each occurrence in the sequence of steps of a proof
cas	Hartley cas function; Eq. (5.59)
$\mathcal{C}f$	Cauchy transform of the function f ; Eq. (3.19)
$C(z)$	Fresnel cosine integral; Eq. (14.171), HT-01
chirp(x)	chirp function, Exercise 18.13
$Ci(x)$	cosine integral; Eq. (8.78), HT-01
ci(x)	cosine integral; HT-01
$cie(\alpha, \beta)$	cosine-exponential integral; HT-01
$Cie(\alpha, \beta)$	cosine-exponential integral; HT-01
$Cl_2(x)$	Clausen function; HT-01
$C_n^\lambda(x)$	Gegenbauer polynomials (ultraspherical polynomials); §9.1, Eq. (11.298), HT-01
D	electric displacement; §19.1
\mathcal{D}	space of all C^∞ functions with compact support; §2.15.2, §10.2
\mathcal{D}'	space of all distributions on \mathcal{D} ; §10.2
\mathcal{D}'_+	space of distributions with support on the right of some point; §10.2
\mathcal{D}_{L^p}	space of test functions; §10.2
\mathcal{D}'_{L^q}	space of distributions; §10.2
$\mathcal{D}'_{\mathcal{L}}$	space of distributions; Eq. (17.240)
$D(x)$	Dawson's integral; Eq. (5.32)
$D_n(\theta)$	Dirichlet kernel; Eq. (6.56)
$D_n^\lambda(x)$	ultraspherical function of the second kind; Eq. (11.299)
$-e$	electronic charge
E	identity element; §2.10, Eq. (2.150)
E	energy of a signal; Eq. (18.1)
E	one-dimensional Euclidean space
E^1	one-dimensional Euclidean space; §2.11.1
E^n	n -dimensional Euclidean space; §2.11.1
E^σ	class of entire functions of exponential type; §2.8.7, §7.4
\mathcal{E}	space of all C^∞ functions with arbitrary support on \mathbb{R} ; §10.2
$\mathcal{E}(t)$	envelope function; Eq. (18.76)
\mathcal{E}'	space of distributions having compact support; §10.2
$E_1(x)$	exponential integral; Eq. (5.98)
E_n	eigenvalues of the unperturbed Hamiltonian; §22.4, Eq. (22.57)
$E_n(z)$	exponential integral; Eq. (14.200), HT-01
$\mathbf{E}(t)$	electric field; §17.9
$\mathbf{E}_i(\omega)$	incident electric field; Eq. (20.3)
$\mathbf{E}_r(\omega)$	reflected electric field; Eq. (20.3)

\mathbf{E}_L	left circularly polarized electric wave; Eq. (21.16)
\mathbf{E}_R	right circularly polarized electric wave; Eq. (21.17)
$Ei(x)$	exponential integral function; Eq. (5.101), HT-01
$\text{erf}(z)$	error function; Eq. (5.27), HT-01
$\text{erfc}(z)$	complementary error function; Eq. (5.141)
$\mathbf{E}_\nu(z)$	Weber's function; HT-01
F	Lorentz force; Eq. (21.50).
$f()$ or f	function (at no particular specified point); §1.2
$f(x)$	function f evaluated at the point x ; §1.2
f_j	oscillator strength; §19.2
$f[n]$	element of a discrete sequence; §13.2, §13.6
$\{f[n]\}$	discrete sequence; §13.6
$f_e(x)$	even function; Eq. (4.8)
$f_o(x)$	odd function; Eq. (4.9)
$f_{\downarrow}(c)$	limit approaching c from $c + 0$; Eq. (2.22)
$f_{\uparrow}(c)$	limit approaching c from $c - 0$; Eq. (2.23)
$\mathcal{F}f$	fourier transform of the function f ; §2.6, Eq. (2.46)
$\mathcal{F}_n f$	n -dimensional Fourier transform of the function f ; §15.6
$\mathcal{F}^{-1}f$	inverse Fourier transform of the function f ; §2.6, Eq. (2.47)
$\mathcal{F}_c f$	Fourier cosine transform of the function f ; Eq. (5.41)
$\mathcal{F}_s f$	Fourier sine transform of the function f ; Eq. (5.40)
\mathcal{F}_N	N -point DFT operator; §13.4
\mathcal{F}_Q	fractional Fourier transform; §18.10, Eq. (18.147)
\mathcal{F}_α	discrete fractional Fourier transform; §18.13, Eq. (18.240)
\hat{f}	Fourier transform of the function f ; §2.6
\tilde{f}	conjugate series of f ; Eq. (6.118); alternative notation for $\mathcal{H}f$; §6.1
f'	derivative of the function f
$f^+(z)$	function f evaluated at an interior point to a contour; Eq. (3.152)
$f^-(z)$	function f evaluated at an exterior point to a contour; Eq. (3.153)
$F_n(\theta)$	Fejér kernel; Eq. (6.63)
${}_1F_1(\alpha; \beta; x)$	Kummer's confluent hypergeometric function; Eq. (5.30), HT-01
${}_2F_1(a, b; c; z)$	hypergeometric function (or Gauss' hypergeometric function); HT-01
F_{ext}	external force; §17.9
F_{rad}	radiative reaction force; §17.9
$f(\omega, 0)$	scattering amplitude at $\theta = 0$; Eq. (19.309)
$F_{\mathbf{h}}$	scattering factor; §23.4
$\text{floor}[x]$	the greatest integer $\leq x$
$G(a, x)$	Hilbert transform of the Gaussian function; §4.7. The abbreviation $G(1, x) \equiv G(x)$ is employed; §9.3
$G_{kl_1 \dots l_2}^{(n)}(t_1, t_2, \dots, t_n)$	tensor components of the n th-order response function; §22.1

\hbar	Planck's constant divided by 2π
H	Hilbert transform operator on \mathbb{R} ; Eqs. (1.2) and (1.4)
\mathcal{H}	Hilbert transform operator for the disc; Eq. (3.202)
\mathbf{H}	magnetic field; Eq. (19.7)
\mathcal{H}_τ	Hilbert transform operator for period 2τ ; Eq. (3.286)
Hf	Hilbert transform of the function f ; §1.2
$(Hf)(x)$	Hilbert transform of the function f on the real line evaluated at the point x ; Eq. (1.2)
$H_e f$	Hilbert transform of the even function f on \mathbb{R}^+ ; Eq. (4.11)
$H_o f$	Hilbert transform of the odd function f on \mathbb{R}^+ ; Eq. (4.12)
$H_1 f$	one-sided Hilbert transform of the function f ; Eq. (8.18)
$H_1 f$	Hilbert's integral of the function f ; Eq. (7.33)
$H_n f$	n -dimensional Hilbert transform of the function f ; Eq. (15.26)
$\mathcal{H}_n f$	general n -dimensional Hilbert transform of the function f in E^n ; Eq. (15.2)
$\mathcal{H}_{n,\varepsilon} f$	general n -dimensional truncated Hilbert transform of the function f in E^n ; Eq. (15.7)
$H_{(k)} f$	Hilbert transform of the function $f(x_1, x_2, \dots, x_k, \dots, x_n)$ in the variable x_k ; Eq. (15.36)
H^{-1}	inverse Hilbert transform operator; Eq. (4.26)
H^+	adjoint of the Hilbert transform operator; Eq. (4.194)
H_α	fractional Hilbert transform operator; Eqs. (18.209) and (18.216)
\mathcal{H}	Hamiltonian for an electronic system; §22.4, Eq. (22.54)
\mathcal{H}_0	unperturbed Hamiltonian for an electronic system; §22.4, Eq. (22.57)
\mathcal{H}	space of test functions; §10.14
\mathcal{H}	inner product space; §2.10.1
\mathcal{H}	Hilbert space; §2.10
$H_n f$	n -dimensional Hilbert transform of the function f , for $n \geq 2$; Eq. (15.26)
$H_1(f, g)(x)$	bilinear Hilbert transform; §16.5
$H_a(f, g)(x)$	bilinear singular integral operator; Eq. (16.85)
$H_n(x)$	Hermite polynomials; §9.3, Eq. (9.39), HT-01
$H(x)$	Heaviside step function; Eqs. (10.54) and (18.116)
$H(\omega)$	response function for a linear system; Eq. (13.1), §18.2
$H_p(\omega)$	fractional Hilbert filter; §18.9, Eq. (18.142)
$H^p(D)$	Hardy space for the unit disc; §2.10.2
H^p	Hardy space for the upper half complex plane; §2.10.2
H_ν	response function at the frequency ν ; Eq. (13.3)
$H_\varepsilon f$	truncated Hilbert transform; Eq. (3.3)
$H_E f$	truncated Hilbert transform; Eq. (4.507)

$H_M f$	maximal Hilbert transform function; Eq. (7.280)
$\mathcal{H}_M f$	maximal Hilbert transform function; Eq. (7.282)
H_{SF}	Hilbert–Stieltjes transform of the function F ; Eq. (4.551)
H_K	Kober’s extension of the Hilbert transform operator; §16.3
H_{R_m}	Redheffer’s extension of the Hilbert transform operator; §16.2
\mathbf{H}_j	vectorial Hilbert transform operator; Eq. (16.100)
$H_{\theta, \varepsilon}$	truncated directional Hilbert transform operator; Eq. (16.103)
H_θ	directional Hilbert transform operator; Eq. (16.104)
H_θ	helical Hilbert transform operator; Eq. (16.131)
H_{M_θ}	directional maximal Hilbert transform operator; Eq. (16.105)
H_{M_θ}	maximal helical Hilbert transform operator; Eq. (16.132)
$H_{M_{n\theta}}$	double maximal helical Hilbert transform operator; Eq. (16.136)
$H_\Gamma f$	Hilbert transform of f along the curve Γ ; Eq. (16.109)
$\bar{H}_\Gamma f$	modified Hilbert transform of f along the curve Γ ; Eq. (16.113)
$H_A f$	Hartley transform of a function f ; Eq. (5.58)
H_A^{-1}	inverse Hartley transform operator; Eq. (5.60)
$H_{\pm v}^{(1)}(z), H_{\pm v}^{(2)}(z)$	Bessel functions of the third kind (Hankel functions of the first kind and second kind, respectively); §9.9
$h_n^{(1)}(z)$	spherical Bessel functions of the third kind; Eq. (9.131) (spherical Hankel functions of the first kind)
$h_n^{(2)}(z)$	spherical Bessel functions of the third kind; Eq. (9.132) (spherical Hankel functions of the second kind)
$h_n(x)$	Hermite–Gaussian functions; Eq. (18.179)
\mathbf{h}_k	discrete Hermite–Gaussian vector functions; Eqs. (18.254) and (18.255)
$\mathbf{H}_v(z)$	Struve’s function; Eq. (9.77), HT-01
$H_D\{f[n]\}$	discrete Hilbert transform of the sequence $\{f[n]\}$; Eq. (13.127)
$\{H_{SD}f\}(x)$	semi-discrete Hilbert transform of the sequence $\{f[\]\}$; Eq. (13.133)
$\mathcal{H}_D\{f[n]\}$	alternative definition of the discrete Hilbert transform; Eq. (13.158)
$(\mathcal{H}_{D_{pq}}x)[n]$	discrete fractional Hilbert transform; Eq. (18.269)
i	imaginary unit (engineers typically use j); §2.8
I	identity operator; §4.4
I	interval; §7.9
$ I $	length of an interval; §7.9
$I_n(x)$	modified Bessel function of the first kind; HT-01
$i(t)$	input (time-dependent in general) to a system; §17.1–17.2
iff	if and only if
Im	imaginary part of a complex function
inf	infimum, the greatest lower bound of a set; §2.8

xxx	<i>Symbols</i>
Ind f	index of a function; Eq. (11.179)
J	SI unit of energy, the joule; §19.1
$J_{\pm\nu}(z)$	Bessel function of the first kind; §9.6, HT-01
$\mathbf{J}_{\nu}(z)$	Anger's function; §9.12, HT-01
$j_n(z)$	spherical Bessel function of the first kind; Eq. (9.115)
k	wave number; Eq. (19.87)
\mathbf{k}	wave vector; §20.7
$k(x, y)$	Kernel function; §1.2, Eq. (1.3)
$K(x)$	Calderón–Zygmund kernel function; §15.1
$K_n(x)$	modified Bessel function of the third kind; HT-01
$l(I)$	length of an interval I ; §2.11.1
$\mathcal{L}f$	Laplace transform of the function f ; Eq. (5.91)
\mathcal{L}_2f	bilateral (or two-sided) Laplace transform of the function f ; Eq. (5.92)
L	class of functions that are Lebesgue integrable on a given interval; §2.11.1
$L(a, b)$	class of functions that are Lebesgue integrable on the interval (a, b) ; 2.11.1
L^1_{loc}	class of functions that are Lebesgue integrable on every subinterval of a given interval; Eq. (4.121)
L^2	class of functions that are Lebesgue square integrable on a given interval; §2.11.1
L^p	class of functions f such that $ f ^p$ is Lebesgue integrable on a given interval; §2.11.1
$L^p(\mathbb{R})$	class of functions f such that $ f ^p$ is Lebesgue integrable on the real line; §2.11.1
l^p	§13.11
$l^p(\mathbb{Z})$	§13.11
L^{∞}	class of essentially bounded functions; §2.11.1
$L^p_{2\pi}$	class of periodic functions f such that $ f ^p$ is Lebesgue integrable on the interval $(0, 2\pi)$. $L^p_{2\tau}$ has a similar meaning for periodic functions with period 2τ .
$L^{\alpha,p}$	class of functions f such that $ x ^{\alpha} f(x) ^p$ is Lebesgue integrable on a particular interval; Eq. (7.186)
$L^p(\mu)$	class of μ -measurable functions; §7.12
$L_n(x)$	Laguerre polynomials; §9.4, Eq. (9.60)
$\mathbf{L}_{\nu}(z)$	modified Struve function; HT-01
$\text{Li}_n(z)$	polylogarithm function; HT-01
$\text{Li}_2(z)$	dilogarithm function; HT-01
Lip m	Lipschitz condition of order m ; §2.3
log	logarithm to the base e ; the alternative notation \ln is also common usage
\log^+f	maximum of $\{\log f , 0\}$; Eq. (7.74)

M	magnetization; Eq. (20.111)
$\text{mod } z$	modulus of a complex number; Eq. (2.68)
$m(E)$	measure of the set E ; §2.11.1
Mf	Hardy–Littlewood maximal function; §7.9
Mf	Mellin transform of f ; Eq. (5.102)
M^{-1}	inverse Mellin transform operator; Eq. (5.107)
m	SI unit for length, the meter; §19.1
$m\{g(\lambda)\}$	distribution function of g ; §4.25, §7.2, Eqs. (4.556), (7.55)
$m_{X,Y}(\omega)$	relative multiplier connecting $X(\omega)$ and $Y(\omega)$; Eq. (18.60)
\mathbb{N}	set of positive integers; 1, 2, 3, . . .
N	complex refractive index; Eq. (19.90)
N^{NL}	nonlinear complex refractive index; §22.13
\mathcal{N}	number of molecules per unit volume; §19.2
$n(\omega)$	angular frequency-dependent refractive index; Eq. (19.91)
$n^{\text{NL}}(\omega, E)$	nonlinear refractive index; §22.13, Eq. (22.238)
$N_{\pm}(\omega)$	complex refractive indices for circularly polarized modes; §21.3, Eq. (21.47)
$n_{\pm}(\omega)$	real parts of $N_{\pm}(\omega)$; Eqs. (21.79) and (21.80)
\mathcal{O}	linear operator on a vector space; §2.10
\mathcal{O}^+	<i>adjoint</i> operator to \mathcal{O} ; §2.10
\mathcal{O}^{-1}	inverse of an operator \mathcal{O} ; §2.10
$O()$	Bachmann order notation, of the order of; Eq. (2.1)
$o()$	Landau order notation, of the order of; Eq. (2.6)
\mathcal{O}'_C	space of distributions that decrease rapidly at infinity; §10.2
$P \int$	Cauchy principal value; §2.4, Eq. (2.18)
$P(r, \theta)$	Poisson kernel for the disc; Eq. (3.49)
$P(x, y)$	Poisson kernel for the half plane; Eq. (3.31)
P_{ε}	Poisson operator; §7.10, Eq. (7.290)
P_+	projection operator; Eq. (4.352)
P_-	projection operator; Eq. (4.353)
$Pf(x^{-1})$	pseudofunction; §10.1
$\mathbf{P}(\mathbf{x})$	electric polarization of a medium; Eq. (19.1)
$P_n(x)$	one of the orthogonal polynomials; §9.1
$P_n(x)$	Legendre polynomials; §9.2, Eqs. (9.10) and (9.27)
$P_v^m(x)$	associated Legendre function of the first kind; HT-01
$P_n^{(\alpha, \beta)}(x)$	Jacobi polynomials; §9.1
\mathcal{P}_{τ}	space of periodic testing functions of period τ ; §10.2
\mathcal{P}'_{τ}	space of periodic distributions of period T ; §10.2
$p.v. \frac{1}{x}$	distribution; §10.1

xxxii

Symbols $\mathcal{P}\frac{1}{x}$

distribution; §10.1

 $Q(r, \theta)$

conjugate Poisson kernel for the disc; Eq. (3.50)

 $Q(x, y)$

conjugate Poisson kernel for the half plane; Eq. (3.32)

 Q_ε

conjugate Poisson operator; §7.10, Eq. (7.291)

 $Q_n(x)$

Legendre function of the second kind; Eq. (11.263)

 $Q_n^m(x)$

associated Legendre function of the second kind; HT-01

 $Q_n^{(\alpha, \beta)}(x)$

Jacobi function of the second kind; HT-01

 R

reflection operator; Eq. (4.73)

 R

radius for a semicircular contour

 A

Radon transform; §5.10, Eqs. (5.152) and (5.155)

 $R_i(z_i)$ residue corresponding to the pole at $z = z_i$; §2.8.5 $R_j f$ Riesz transform of the function f ; §15.12 \mathbb{R}

real line; the set of real numbers

 \mathbb{R}^+

positive real axis interval; §3.4

 $\mathbb{R} \times \mathbb{R}$

Euclidean plane

 \mathbb{R}^n n -dimensional Euclidean space; §2.15.2 \mathcal{R}

simply connected region; §2.8.1

 \mathcal{R}

radius for a semicircular contour

 \mathfrak{R}_p

Riesz constant; §4.20, Eqs. (4.382) and (4.384)

 $\tilde{r}(\omega)$

generalized or complex reflectivity; Eq. (20.1)

 $\tilde{r}_\pm(\omega)$

generalized reflectivity for circularly polarized modes; Eq. (21.132)

 $r(\omega)$

reflectivity amplitude; Eq. (20.1)

 $r(t)$

response (time-dependent) from a system; §17.1, §17.2

 R_{n0}

rotational strength; Eq. (21.233)

 $R(\omega)$

reflectivity; Eq. (20.2)

 $re^{i\theta}$ polar form of the complex number z Re

real part of a complex number

 $\operatorname{rect}(x)$

rectangular pulse function; §18.7.3

 $\operatorname{Res}\{g(z)\}_{z=z_0}$ residue at the pole $z = z_0$ of the function g ; §2.8.5, Eq. (2.93) Sf Stieltjes transform of the function f ; Eqs. (5.77), and (8.6) S_a

dilation operator (homothetic operator); Eqs. (4.70) and (15.68)

 sgn

signum function (sign function); Eqs. (1.14) and (18.120)

 S^{n-1} locus of points $x \in \mathbb{R}^n$ for which $|x| = 1$; §16.6 $S(z)$

Fresnel sine integral; Eq. (14.172), HT-01

 $S(E)$ S -function (S -matrix); §17.12 $S(\omega)$ Fourier transform of a signal $s(t)$ in the frequency domain; §18.1, Eq. (18.2) $s(t)$

signal in the time domain; §18.1

 $\operatorname{Shi}(z)$

hyperbolic sine integral function; Eq. (14.201), HT-01

 $\operatorname{Si}(x)$

sine integral; Eq. (8.79), HT-01

	<i>Symbols</i>	xxxiii
$\text{si}(x)$	integral; Eq. (9.170), HT-01	
$\text{sie}(\alpha, \beta)$	sine-exponential integral; HT-01	
$\text{sinc } x$	sinc function; Eq. (4.260), HT-01	
sup	supremum, the least upper bound	
supp	support of the function; §2.15.2	
T	finite Hilbert transform operator; chap. 11, Eq. (11.2)	
T	used to denote a distribution; §2.15.2	
T_{ab}	finite Hilbert transform operator on the interval (a, b) ; Eq. (12.98)	
$T_n(x)$	Chebyshev polynomials of the first kind; §9.1, HT-01	
\mathbb{T}	circle group; §3.10	
Tr	trace; §22.4, Eq. (22.63)	
$U_n(x)$	Chebyshev polynomials of the second kind; §9.1, HT-01	
$u[n]$	unit step sequence; Eq. (13.91)	
V	total variation of a function; Eq. (4.554)	
V	SI unit for potential, the volt; §19.1	
$w(x)$	weight function; §9.1	
$W(x)$	weight function; §14.4	
w_i	weight points in a quadrature scheme; Eq. (14.15)	
$W^{p,m}$	Sobolev space; §10.2	
\bar{x}_j	any value in the interval $[x_{j-1}, x_j]$; §2.11	
x_i	sampling points in a quadrature scheme; Eq. (14.15)	
$ x $	norm of x in E^n ; §15.1	
\mathbf{x}	vector cross product	
\times	direct product; §10.6. Also used for Cartesian product of Euclidean spaces; §15.13	
$\mathbf{x}(t)$	time-dependent particle displacement; §17.2, §17.9	
$X(z)$	Z transform (one-sided or two-sided); Eqs. (13.38) and (13.39)	
$Y_\nu(z)$	Bessel function of the second kind (Weber's function, Neumann's function); §9.6, 9.8, HT-01	
$y_n(z)$	spherical Bessel function of the second kind; Eq. (9.116)	
z	complex variable, $z = x + iy$; Eq. (2.67)	
\bar{z}	complex conjugate of z	
z^*	complex conjugate of z	
z_1	inverse point (or image point) of z ; Eq. (3.35)	
\mathbb{Z}	set of integers $0, \pm 1, \pm 2, \dots$	
\mathbb{Z}^+	set of non-negative integers $0, 1, 2, \dots$	
$Z\{x_n\}$	Z transform of the sequence $\{x_n\}$; §13.6, Eq. (13.38)	