1

Introduction

1.1 Some common integral transforms

Transform techniques have become familiar to recent generations of undergraduates in various areas of mathematics, science, and engineering. The principal integral transform that is perhaps best known is the Fourier transform. The jump from the time domain to the frequency domain is a characteristic feature of a number of important instrumental methods that are routinely employed in many university science departments and industrial laboratories. Fourier transform nuclear magnetic resonance spectroscopy (acronym FTNMR) and Fourier transform infrared spectroscopy (FTIR) are two extremely significant techniques where the Fourier transform methodology finds important application. Two transforms derived from the Fourier transform, the Fourier sine and Fourier cosine transforms, also find wide application. The Laplace transform is often encountered fairly early in the undergraduate mathematics curriculum, because of its utility in aiding the solution of certain types of elementary differential equations. The transforms that bear the names of Abel, Cauchy, Mellin, Hankel, Hartley, Hilbert, Radon, Stieltjes, and some more modern inventions, such as the wavelet transform, are much less well known, tending to be the working tools of specialists in various areas. The focus of this work is about the Hilbert transform. In the course of discussing the Hilbert transform, connections with some of the other transforms will be encountered, including the Fourier transform, the Fourier sine and Fourier cosine offspring, and the Hartley, Laplace, Stieltjes, Mellin, and Cauchy transforms. The Z-transform is studied as a prelude to a discussion of the discrete Hilbert transform.

In this chapter the principal objective is to provide a non-rigorous introduction to the Hilbert transform, and to establish the idea of the Hilbert transform operator. Some brief historical comments are presented on the emergence of the Hilbert transform. Finally, some areas are given where the Hilbert transform finds application.

1.2 Definition of the Hilbert transform

Many of the common integral transforms can be written in the following form:

$$g(x) = \int_a^b k(x, y) f(y) \mathrm{d}y, \tag{1.1}$$

where $k(x, y)$ is called the *kernel function*, or just the kernel of the equation. Equation (1.1) can also be thought of as an example of an *integral equation*, if one desires to determine the function f in terms of g. More specifically, it is termed a Fredholm equation of the first kind. The limits on the integral can be finite or infinite. When the kernel function has a singularity in the integration range, it is possible in a number of cases to extend the definition of the integral in Eq. (1.1) to accommodate these cases. Such equations are referred to as singular integral equations.

The Hilbert transform on $\mathbb{R}$, the real line, is defined by

$$Hf(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y)\mathrm{d}y}{x - y}, \quad \text{for } x \in \mathbb{R}. \tag{1.2}$$

The kernel function in this definition is given by

$$k(x, y) = \frac{1}{\pi(x - y)}, \tag{1.3}$$

which is singular when $y = x$. The symbol $P\int$ denotes an extension of the normal definition of the integral called the *Cauchy principal value*. This is discussed in detail in Chapter 2. The integral becomes well behaved for many common functions if an infinitesimally small section of the integration interval centered at the singularity $y = x$ is deleted, as part of the definition of the integral. This is the essential effect of evaluating the integral as a principal value integral.

A word on notation may be useful at this juncture. Commonly, f is used to denote a function of a single variable and $f(x)$ is the value of the function evaluated at the point x. It is prevalent in the sciences to use the notation $f(x)$ to denote the function and also the value of the function evaluated at the point x. Usually the context makes it clear which of the two meanings is intended, although the use of f or $f(\ )$ for the function, and $f(x)$ for the value of the function evaluated at the point x, makes the meaning much clearer. The interpretation of Eq. (1.2) is that Hf signifies a new function and $Hf(x)$ is the value of this function evaluated at the point x. The notation Hf is used when there is no need to specify the point at which the transform is evaluated, which is convenient in a number of cases, particularly where additional operators such as the Fourier or inverse Fourier transform operator are also being applied to the function f. Occasionally the notation $H[\ ]$ or $H\{\ \}$ is employed; this is expedient when the Hilbert transform of a product of functions is taken, but the notation is not used exclusively for this purpose. In this book the notation $H[f(x)]$ or $H\{f(x)\}$ is employed with some frequency as a shorthand for $H[f(t)](x)$. In the latter form, t is the dummy integration variable for the Hilbert transform, and the function Hf is evaluated at the point x. Occasionally the notation $H[f, x]$ is used in the literature to denote the Hilbert transform of the function f evaluated at the point x. Sometimes, mostly by mathematicians, the Hilbert transform of the function

f is denoted by $\tilde{f}$. In the literature, the symbol T is also employed to denote the Hilbert transform. In this book, T is used to denote the finite Hilbert transform. When no confusion is likely, operator identities involving H are written with no function specified, and it is assumed that functions can be found for which the operator equality holds.

Historically, Eq. (1.2) was not the definition given by David Hilbert. Working in the area of integral equations, he arrived at a pair of integral equations connecting the real and imaginary parts of a function analytic in the unit disc, leading to the definition of the Hilbert transform for the circle (Hilbert, 1904, 1912). The transform appearing in Eq. (1.2) seems to have been first discussed with some level of rigor by the cricket loving English mathematician G. H. Hardy (1902, 1908), and named by him in 1924 the Hilbert transform, in honor of Hilbert's contribution. It is perhaps interesting to speculate how this transform might have been named by later workers had Hardy not graciously named the transform as he did. In a sense, Alfred Tauber's contribution (Tauber, 1891) appears to have been overlooked. In hindsight, perhaps the transform should bear the names of the three aforementioned authors. Most of the early developments on the Hilbert transform were not performed by David Hilbert, but by Hardy (1924a, 1924b, 1932) and Titchmarsh (1925a, 1929, 1930a, 1930b). A related form was given by Young (1912). Variants of the Hilbert transform on $\mathbb{R}$ are presented in later chapters; these include the Hilbert transform for the circle, the finite Hilbert transform, the multi-dimensional Hilbert transform, the discrete Hilbert transform, and others.

The reader is alerted to the existence of an alternative definition of the Hilbert transform for the real line, one where the kernel $k(x, y) = \{\pi(y - x)\}^{-1}$ is employed. Unfortunately, a consensus agreement on the definition has not been reached, and both forms occur rather commonly in the literature, though the definition given in Eq. (1.2) appears to be increasingly favored. For a number of purposes this difference in sign is not important, but obviously is significant for the evaluation of the Hilbert transform of a particular function, which means that the reader needs to be alert to the sign choice when pulling entries from tables of Hilbert transforms. Occasionally the Hilbert transform is defined with the factor π^{-1} omitted. Employing the definition given in Eq. (1.2) does have the advantage that factors of π that would frequently appear are incorporated into the definition of the Hilbert transform. A few authors define the Hilbert transform with the imaginary unit factor included, that is, π^{-1} is replaced by $(\pi \mathrm{i})^{-1}$.

Note that nothing has been said about what conditions must be specified for the function f in order that the integral in Eq. (1.2) exists. Different levels of rigor can be brought to bear on this question. For almost all applications in the physical sciences, the existence of the Riemann integral of the function $|f|^2$ over the interval $(-\infty, \infty)$ is all that is required to guarantee that the Hilbert transform of f is bounded. The Hilbert transform can be defined for a wider class of functions than the aforementioned, and this is addressed in Chapter 3.

1.3 The Hilbert transform as an operator

The key idea in the application of any of the simple integral transforms is that the function f is acted on by an "integral operator," to yield a new function, g, which is referred to as the "name" transform of f. In the case of the Hilbert transform, the integral operation is given by

$$H \equiv \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(\,) - s}, \tag{1.4}$$

where the identity of the function and the point at which the Hilbert transform is evaluated are left unspecified. The Hilbert transform of f can be thought of as the application of the integral operator in Eq. (1.4) on the function $f(\,)$, to yield

$$Hf(\,) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(\,) - s} f(s), \tag{1.5}$$

and the left-hand side of Eq. (1.5) is frequently denoted by the function $g(\,)$. Clearly, the function g depends on the entire shape of f. In other words, g at some point x, $g(x)$, is not determined simply by the value of the function f evaluated at the same point. That is, g has a *non-local* dependence on f. The situation where $g(x)$ is determined directly by the value $f(x)$ arises when there is a simple functional connection between f and g; for example, suppose $g(x) = \sin[f(x)]$, then the value of g at the point x depends only on the value of f evaluated at x. This notion has important consequences. A function f could be zero over a large region of the real axis and finite for a small region, but its Hilbert transform could be everywhere non-zero. Applications will be encountered later that reflect this type of behavior.

To visualize the changes that take place when the Hilbert transform of a function is evaluated, consider the following choice:

$$f(x) = \frac{a}{a^2 + x^2}, \tag{1.6}$$

where a is a real positive constant. This functional form appears in several diverse applications, and is sometimes referred to as a Cauchy pulse, and in other applications is closely related to the Lorentzian profile. The Hilbert transform of this function is given by

$$g(x) = Hf(x) = \frac{x}{a^2 + x^2}. \tag{1.7}$$

Figure 1.1 shows a plot of $f(x)$ and its Hilbert transform for the value $a = 1$.

The particular methods that are most effective for evaluating this relatively straightforward Hilbert transform are discussed in Chapter 2 and illustrated with examples in Chapters 3 and 4.

The function f of the preceding example can be recovered from g using the expression $f(x) = -Hg(x)$. In fact, this is a rather general result. The two formulas

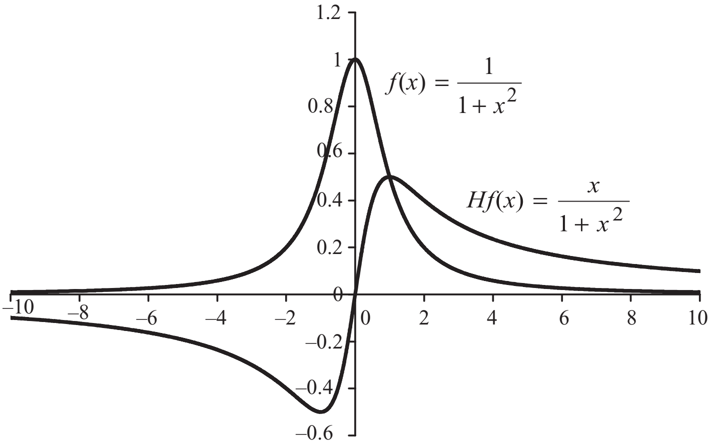

Figure 1.1. Plot of the Cauchy pulse and its Hilbert transform.

$g(x) = Hf(x)$ and $f(x) = -Hg(x)$ constitute a *Hilbert transform pair*. This Hilbert transform pair is explored in detail in later chapters, and it is shown that there is a very close connection with the theory of analytic functions. Pairs of functions that satisfy this type of skew-reciprocal character have been known for a considerable time. For example, the results (for $a > 0$)

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\sin as \, \mathrm{d}s}{x - s} = -\cos ax \tag{1.8}$$

and

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\cos as \, \mathrm{d}s}{x - s} = \sin ax \tag{1.9}$$

were given well over one hundred years ago (Schlömilch, 1848 p. 153; Bierens de Haan, 1867). The sine and cosine functions thus form a Hilbert transform pair.

Hardy (1908, 1924a, 1924b, 1928a, 1932) was one of those who pioneered the study of the mathematical foundations of the Hilbert transform. Prior to Hilbert's publications, Hardy (1902) had investigated the properties of Cauchy principal value integrals, and, in particular, he derived the preceding two formulas. Let $I(x, a)$ denote the following integral:

$$I(x, a) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\sin as \, \mathrm{d}s}{x - s}, \tag{1.10}$$

where a is a constant. From the preceding formula, Hardy obtained the following differential equation:

$$\frac{\mathrm{d}^2 I}{\mathrm{d}x^2} + a^2 I = 0. \tag{1.11}$$

The topic of differentiation of the Hilbert transform is discussed in detail later. The solution of Eq. (1.11) is

$$I(x, a) = \alpha \cos ax + \beta \sin ax, \tag{1.12}$$

where α and β are arbitrary constants. Setting $x = 0$ and using the result

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as \, ds}{s} = \operatorname{sgn} a, \tag{1.13}$$

where

$$\operatorname{sgn} a = \begin{cases} 1, & a > 0 \\ 0, & a = 0 \\ -1, & a < 0 \end{cases} \tag{1.14}$$

gives $\alpha = -\operatorname{sgn} a$. It is straightforward to show that $I(x, a) = -I(-x, -a)$, from which it follows that $\beta = 0$, and hence

$$I(x, a) = -\operatorname{sgn} a \cos ax. \tag{1.15}$$

Hardy (1902) gave this result for the case $a > 0$, and he also gave a result equivalent to Eq. (1.9).

1.4 Diversity of applications of the Hilbert transform

Historically, work on Hilbert transforms developed on three main fronts. Mathematicians made the seminal developments in the first quarter of the twentieth century by putting the Hilbert transform into various useful forms, and established a number of key theorems that turned out to be of critical importance for future developments in the physical sciences. Hilbert transforms arose first in potential theory. Around the time of the dawn of modern quantum theory, Kramers (1926, 1927) and, working independently, Kronig (1926) obtained the reciprocal relations between the frequency dependent refractive index and the absorption coefficient of a medium. The resulting equations involved principal value integrals over the frequency interval $[0, \infty)$, which can be recast as a pair of standard Hilbert transforms. These equations became known in the physics and chemistry literature as the Kramers–Kronig relations. In parallel with this development, electrical engineers applied the same and some closely related mathematical ideas in circuit analysis (Carson, 1926). The real and imaginary parts of the general complex impedance were found to be connected to each other by Hilbert transforms. These relations are sometimes referred to as the Bode relations (Bode, 1945). In branches of engineering the Hilbert transforms are sometimes referred to as Wiener–Lee transforms (Papoulis, 1962, p. 192). In modern signal processing the terms 90° phase shift filter or quadrature filter are also employed to describe a Hilbert transform. The former of these two designations comes from the fact that the Hilbert transform of a sine function yields a cosine function, and this can be recast as a sine function with a shift of the argument by 90°. Somewhat later, with activity rising significantly in the early 1960s, Hilbert transforms found important applications in the study of various scattering processes in elementary particle physics and some other branches of physics. The key equations developed to describe the scattering processes are called *dispersion relations*, which are, in many cases, Hilbert transform relations or relatively minor extensions of the Hilbert transform concept. The Hilbert transform technique has clearly acquired multiple names as it has been employed in different applications. This multiplicity of names makes it more difficult to assess the true impact of Hilbert's contribution to transform calculus in the physical sciences. In addition to Hilbert, perhaps it is not inappropriate to give due credit to the nineteenth century mathematicians Poisson and, in particular, Cauchy, whose contributions laid the foundations for the work of Hilbert and others on the transform that finds such a diverse number of applications.

The question of why one should be interested in studying the theory of Hilbert transforms can be best answered in the following manner. There are numerous practical applications of Hilbert transforms, such as those mentioned in the preceding paragraph. To that list of applications can be added problems in aerofoil theory, crack formation in materials, aspects of the theory of elasticity, applications in wave propagation theory, problems in potential theory, and the study of dispersion forces. Further applications arise in certain areas in digital signal processing, and problems in the reconstruction of images. Readers with an interest in the stock market might be fascinated to see how a discrete version of the Hilbert transform has been used as a modeling tool (Ehlers, 2001). For some of these topics, the Hilbert transform or some variant of the standard form occurs as part of an integral equation or of an integro-differential equation. An example that is discussed later is the study of solitary waves. Because of the rich and diverse array of applications, the study of Hilbert transform theory can be a rewarding exercise.

Hilbert transform theory of course finds a number of applications in pure mathematics. The theory of the conventional Hilbert transform can be viewed as a paradigm for the mathematical investigation of singular integrals in general. This opens up a whole area of study in singular integral equations. Hilbert transform theory has served as a springboard to the study of singular integrals in n-dimensional Euclidean space. The Hilbert transform has played an important role in addressing some fundamental questions in the theory of Fourier series. This transform has a very close connection to some areas of complex analysis, and it plays an essential role in the theory of Fourier transforms of causal functions. The Hilbert transform is the key ingredient in characterizing operators that commute with the translation and dilation operators. Parts of all of the aforementioned topics are discussed in an introductory fashion in the following chapters.

Notes

The end-notes for each chapter provide sources, both books and journal articles, where additional reading on various topics may be pursued. The books that are recommended on standard topics reflect in large part the contents of the author's personal library. On many standard topics, particularly the background material covered in Chapter 2, the reader should be able to find a large number of additional reference texts beyond the ones cited. For a delightful account of the life and times of David Hilbert, intended for a general audience, see Reid (1996).

§1.1 For further reading on integral equations, consult Gakhov (1966), Hochstadt (1973), Tricomi (1985), Mikhlin and Prössdorf (1986), Pipkin (1991), Muskhelishvili (1992), and Kress (1999). Good sources on integral transforms with an applied emphasis include Sneddon (1972) and Davies (1978). The books by Zayed (1996) and Debnath and Bhatta (2007), and the individual accounts in Poularikas (1996a), are highly recommended reading.

§1.2 Hardy's work referenced in this book can be found in the seven volumes of his collected papers, Hardy (1966).

§1.3 Additional Hilbert transform pairs can be found in the nineteenth century literature; see, for example, Schlömilch (1848) or Bierens de Haan (1867). For some more recent collections of Hilbert transforms, see the following: Erdélyi *et al*. (1954, Vol. II), MacDonald and Brachman (1956), Smith and Lyness (1969), Alavi-Sereshki and Prabhakar (1972), and Hahn (1996a, 1996b). Hilbert transform relations of the type given in Eqs. (1.8) and (1.9) are due to the great French mathematician Augustin-Louis Cauchy.

§1.4 Some further reading on various applications of the Hilbert transform can be found in: Tricomi (1985, p. 173) and Zayed (1996, p. 287) for aerofoil theory; Wright and Hutchinson (1999) for the determination of oscillator phases for atomic motions; Ferry (1970), Booij and Thoone (1982), Madych (1990), and Herdman and Turi (1991), for elasticity theory; Aki and Richards (1980, p. 852) for crack propagation; Hinojosa and Mickus (2002) for the study of gravity gradient profiles; Červený and Zahradník (1975) for a review of geophysical applications; Weaver and Pao (1981), Beltzer (1983), and Bampi and Zordan (1992), for wave propagation theory; Duffin (1972), Nabighian (1984), and Sugiyama (1992), for aspects of potential theory; Sakai and Vanasse (1966) for an application in Fourier spectroscopy; Karl (1989), Hahn (1996a, 1996b), Oppenheim, Schafer, and Buck, (1999), for signal processing; and Lowenthal and Belvaux (1967), Herman (1980), Kohlmann (1996), Arnison *et al*. (2000), Davis, McNamara, and Cottrell (2000), and Shaik and Iftekharuddin (2003), for image reconstruction theory. A study of dispersion forces using dispersion theoretic techniques can be found in Feinberg, Sucher, and Au (1989). A number of applications have been made in Raman spectroscopy; see Chinsky *et al*. (1982), Stallard *et al*. (1983), Patapoff, Turpin, and Peticolis (1986), and Lee and Yeo (1994). For the development of a dispersion-type relation for the ground-state energy of two-electron atomic systems as a function of nuclear charge, see Ivanov and Dubau (1998). For further general reading on matters mathematical, see Butzer and Trebels (1968), Butzer and Nessel (1971), and Pandey (1996). A concise introductory account on the Hilbert transform can be found in Peters (1995).

Exercises

The table of Hilbert transforms in Appendix 1 should prove to be of value to you, both for checking the answers to a number of exercises throughout the book, and for solving some of the exercises.

1.1 Given

$$I(x,a) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\cos as\, \mathrm{d}s}{x-s},$$

where a is a constant, set up a differential equation by differentiation with respect to x, and hence determine the value for $I(x,a)$. Justify the differentiation step.

1.2 Given

$$P \int_{-\infty}^{\infty} f(s)\mathrm{d}s = \lim_{\varepsilon \to 0+} \left\{ \int_{-\infty}^{x-\varepsilon} f(s)\mathrm{d}s + \int_{x+\varepsilon}^{\infty} f(s)\mathrm{d}s \right\},$$

show for

$$f(s) = (x-s)^{-1} \text{ that } P \int_{-\infty}^{\infty} f(s)\mathrm{d}s = 0.$$

1.3 What is the value of $\int_{-\infty}^{\infty} \mathrm{d}s/(x-s)$?

1.4 Show that $Hf(x)$ equals $-a/(x^2+a^2)$ for $f(s) = s/(a^2+s^2)$, where a is a positive constant. Hint: The identity

$$\frac{s}{(s^2+a^2)(x-s)} = \frac{1}{x^2+a^2} \left\{ \frac{x}{x-s} + \frac{xs}{s^2+a^2} - \frac{a^2}{s^2+a^2} \right\}$$

leads to a straightforward calculation.

1.5 Show that $Hf(x)$ equals $x/a(x^2+a^2)$ for $f(s) = 1/a^2+s^2$, where a is a positive constant. Hint: The identity

$$\frac{1}{(s^2+a^2)(x-s)} = \frac{1}{x^2+a^2} \left\{ \frac{x}{s^2+a^2} + \frac{1}{x-s} + \frac{s}{s^2+a^2} \right\}$$

simplifies the calculation.

1.6 If c is a constant, evaluate $H[c]$.

1.7 Evaluate $H[\sin(ax+b)]$, where a and b are real constants.

1.8 Evaluate $H[\cos(ax+b)]$, where a and b are real constants.

1.9 Evaluate $H[\sin^2(\alpha x)]$, where α is a real constant.

1.10 If α is a real constant, does $H[x^{-1}\sin(\alpha x)]$ converge?

1.11 For α a real constant, determine whether $H[x^{-1}\cos(\alpha x)]$ converges.

1.12 Prove Eq. (1.13).

1.13 For $f(x) = x(x^2 + \alpha^2)^{-1}$ with α a real constant greater than zero, how does Hf behave as $\alpha \to 0+$?

1.14 If

$$f(x) = \begin{cases} 0, & x < 0 \\ e^{-\alpha x}, & x \geq 0 \end{cases} \quad \text{with } \alpha > 0,$$

evaluate $Hf(x)$.

1.15 If $f(x) = \begin{cases} 0, & |x| > 1 \\ 1, & |x| \leq 1 \end{cases}$, evaluate $Hf(x)$.

1.16 For $a > 0$, is the statement $H[x^2(a^2 + x^2)^{-1}] = -ax\,(a^2 + x^2)^{-1}$, true or false?