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978-0-521-88720-5 - Kazhdan's Property (T)
Bachir Bekka, Pierre de la Harpe and Alain Valette
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Kazhdan's Property (T)

Property (T) is a rigidity property for topological groups, first formulated by D. Kazhdan in the mid-1960s with the aim of demonstrating that a large class of lattices are finitely generated. Later developments have shown that Property (T) plays an important role in an amazingly large variety of subjects, including discrete subgroups of Lie groups, ergodic theory, random walks, operator algebras, combinatorics, and theoretical computer science.

This monograph offers a comprehensive introduction to the theory. It describes the two most important points of view on Property (T): the first uses a unitary group representation approach, and the second a fixed point property for affine isometric actions. Via these the authors discuss a range of important examples and applications to several domains of mathematics. A detailed appendix provides a systematic exposition of parts of the theory of group representations that are used to formulate and develop Property (T).

BACHIR BEKKA is Professor of Mathematics at the Université de Rennes 1, France.

PIERRE DE LA HARPE is Professor of Mathematics at the Université de Genève, Switzerland.

ALAIN VALETTE is Professor of Mathematics at the Université de Neuchâtel, Switzerland.

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Symbols

$U(\mathcal{H})$	unitary group of a Hilbert space \mathcal{H} , page 27
1_G	unit representation of a topological group G , page 28
$\kappa(G, Q, \pi)$	Kazhdan constant, page 29
$\kappa(G, Q)$	Kazhdan constant, page 30
\mathcal{H}^G	subspace of G -invariant vectors in \mathcal{H} , page 31
\subset	sign for subrepresentation, page 32
δ_x	Dirac measure at a point x , page 37
F_k	non-abelian free group on k generators, page 39
$SL_n(\mathbf{K})$	special linear group, page 40
$\mathcal{L}(\mathcal{H})$	algebra of bounded operators on a Hilbert space, page 40
$E_{ij}(x)$	elementary matrix, page 44
$Sp_{2n}(\mathbf{K})$	symplectic group, page 50
$S^{2*}(\mathbf{K}^2)$	space of symmetric bilinear forms on \mathbf{K}^2 , page 51
$S^2(\mathbf{K}^2)$	second symmetric power of \mathbf{K}^2 , page 51
$\text{rank}_{\mathbf{K}} \mathbb{G}$	\mathbf{K} -rank of an algebraic group \mathbb{G} , page 58
$\mathbb{G}(\mathbf{K})$	group of \mathbf{K} -rational points of an algebraic group \mathbb{G} , page 58
$HS(\mathcal{H})$	space of Hilbert–Schmidt operators on a Hilbert space \mathcal{H} , page 65
\tilde{G}	universal covering group of G , page 67
$M_{n,m}(\mathbf{K})$	space of $(n \times m)$ matrices over \mathbf{K} , page 68
$M_n(\mathbf{K})$	algebra of $(n \times n)$ matrices over \mathbf{K} , page 68
$H_{2n+1}(\mathbf{K})$	Heisenberg group, page 69
$\mathcal{O}(\mathcal{H}^0)$	orthogonal group of a real Hilbert space \mathcal{H}^0 , page 74
$Z^1(G, \pi)$	space of 1-cocycles with coefficients in a representation π , page 76
$B^1(G, \pi)$	space of 1-coboundaries in $Z^1(G, \pi)$, page 76
$H^1(G, \pi)$	1-cohomology group with coefficients in a representation π , page 76

List of symbols

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$\text{Hom}(G, H)$	space of continuous homomorphisms from G to H , page 77
$X = (V, \mathbb{E})$	graph with vertex set V and edge set \mathbb{E} , page 81
$\mathcal{H}_{\mathbf{R}}$	complex Hilbert space \mathcal{H} viewed as a real Hilbert space, page 88
$\pi_{\mathbf{R}}$	unitary representation π viewed as an orthogonal representation, page 88
$O(n, 1)$	orthogonal group of the form $-x_{n+1}^2 + \sum_{i=1}^n x_i^2$ over \mathbf{R}^{n+1} , page 93
$\mathbf{H}^n(\mathbf{R})$	real hyperbolic space, page 93
$U(n, 1)$	orthogonal group of the form $- z_{n+1} ^2 + \sum_{i=1}^n z_i ^2$ on \mathbf{C}^{n+1} , page 100
$Sp(n, 1)$	orthogonal group of the form $- z_{n+1} ^2 + \sum_{i=1}^n z_i ^2$ on \mathbf{H}^{n+1} , page 100
$\mathbf{H}^n(\mathbf{C})$	complex hyperbolic space, page 100
$\mathbf{H}^n(\mathbf{H})$	quaternionic hyperbolic space, page 100
$\Gamma \wr H$	wreath product of the groups Γ and H , page 105
$\text{Diff}_+^{1+\alpha}(\mathbf{S}^1)$	a diffeomorphism group of the circle, page 107
$\text{Homeo}_+(\mathbf{R})$	group of orientation preserving homeomorphisms of \mathbf{R} , page 116
$C_c(G//K)$	convolution algebra of continuous K -bi-invariant functions on G with compact support, page 152
$M(G//K)$	algebra of K -bi-invariant compactly supported probability measures on G , page 154
$R[t]$	ring of polynomials over a ring R , page 201
$R[[t^{-1}]]$	ring of formal power series in t^{-1} over R , page 201
G^X	group of continuous mappings from X to G , page 210
LG	loop group of G , page 210
$\Omega_{\mathcal{H}}^k(X)$	Hilbert spaces associated to a random walk, page 219
$\mathcal{E}_{\pi}^k(X)$	Hilbert spaces associated to G -equivariant random walk, page 227
∂A	boundary of a subset A of a graph, page 254
$h(\mathcal{G})$	expanding constant of a graph \mathcal{G} , page 254
$C(G)_{1,+}$	continuous non-negative on G of integral one, page 262
$L_0^2(\Omega)$	space of square-integrable functions on Ω with zero mean, page 265
(ξ, η)	inner product of two vectors, page 289
$\pi_1 \simeq \pi_2$	equivalent representations, page 290
$\rho \subset \pi$	representation ρ contained in π , page 291
\widehat{G}	unitary dual (dual group, in case G is abelian), page 292
\mathbf{S}^1	circle group, page 293

$\pi(G)'$	commutant of $\pi(G)$, page 296
$\mathcal{B}(X)$	σ -algebra of Borel subsets in a topological space X , page 299
$C_c(X)$	space of complex-valued functions with compact support on a topological space X , page 300
\check{f}	$\check{f}(x) = f(x^{-1})$, page 303
${}_a f$	${}_a f(x) = f(ax)$ for a function f on a group G and $a \in G$, page 305
f_a	$f_a(x) = f(xa)$ for a function f on a group G and $a \in G$, page 305
λ_G	left regular representation of a locally compact group G , page 305
$\lambda_{G/H}$	quasi-regular representation of a locally compact group G on $L^2(G/H)$, page 331
P	Poincaré half-plane, page 331
H	Hamiltonian quaternions, page 337
$C(X)$	space of complex-valued continuous functions on a topological space X , page 342
$f * g$	convolution product of two functions f and g on a locally compact group, page 352
\tilde{f}	$\tilde{f}(x) = \overline{f(x^{-1})}$, page 353
$C_0(X)$	space of complex-valued continuous functions vanishing at infinity on a topological space X , page 353
$\mathcal{P}(G)$	convex cone of functions of positive type on a topological group G , page 357
$\mathcal{P}_1(G)$	convex set of normalised functions of positive type on G , page 357
$\text{ext}(\mathcal{P}_1(G))$	extreme points of $\mathcal{P}_1(G)$, page 358
$L^\infty(G)$	space of complex-valued locally measurable functions on G which are bounded locally almost everywhere, page 359
f^*	$f^*(x) = \Delta_G(x^{-1})\overline{f(x^{-1})}$, page 360
$L^1(G)$	Banach $*$ -algebra of absolutely integrable complex-valued functions on a locally compact group, page 360
$\mathcal{P}_{\leq 1}(G)$	convex set of $\varphi \in \mathcal{P}(G)$ with $\varphi(e) \leq 1$, page 360
$\Delta(A)$	character space of a Banach algebra A , page 370
$M(G)$	Banach- $*$ -algebra of finite complex regular measures on G , page 370
$d\mu^*(x)$	defined for a measure μ on a group by $d\mu^*(x) = \overline{d\mu(x^{-1})}$, page 370
$\hat{\mu}$	Fourier transform of a measure $\mu \in M(G)$, page 370

List of symbols

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$C^b(X)$	space of complex-valued bounded continuous functions on a topological space X , page 370
\widehat{f}	Fourier transform of a function $f \in L^1(G)$, page 371
$M_{\leq 1}(G)$	space of positive Borel measures μ on G with $\mu(G) \leq 1$, page 372
$\text{Proj}(\mathcal{H})$	set of orthogonal projections on subspaces of a Hilbert space \mathcal{H} , page 373
\mathbf{Q}_p	field of p -adic numbers, page 377
\mathbf{Z}_p	ring of p -adic integers, page 378
$k((X))$	field of Laurent series, page 378
$\text{Ind}_H^G \sigma$	induced representation, page 388
\prec	sign for weak containment of unitary representations, page 396
π_{univ}	universal representation of a locally compact group, page 412
$C^*(G)$	maximal C^* -algebra of a locally compact group, page 412
$C_{\text{red}}^*(G)$	reduced C^* -algebra of a locally compact group, page 413