Cambridge University Press
978-0-521-88720-5 - Kazhdan's Property (T)
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Introduction

In the mid-1960s, D. Kazhdan defined his *Property (T)* for locally compact groups and used it as a tool to demonstrate that a large class of lattices are finitely generated. Recall that a lattice Γ in a locally compact group G is a discrete subgroup such that the quotient space G/Γ carries a G-invariant probability measure; arithmetic and geometry provide many examples of countable groups which are lattices in semisimple groups, such as the special linear groups $SL_n(\mathbf{R})$, the symplectic groups $Sp_{2n}(\mathbf{R})$, and various orthogonal or unitary groups. Property (T) was defined in terms of unitary representations, using only a limited representation theoretic background. Later developments showed that it plays an important role in many different subjects.

Chapter 1 begins with the original definition of Kazhdan:

> A topological group G has *Property (T)* if there exist a compact subset Q and a real number $\varepsilon > 0$ such that, whenever π is a continuous unitary representation of G on a Hilbert space $\mathcal{H}$ for which there exists a vector $\xi \in \mathcal{H}$ of norm 1 with $\sup_{q\in Q} \|\pi(q)\xi - \xi\| < \varepsilon$, then there exists an invariant vector, namely a vector $\eta \neq 0$ in $\mathcal{H}$ such that $\pi(g)\eta = \eta$ for all $g \in G$.

We discuss some of its first implications, the trivial examples (which are the compact groups), and the following three main ingredients of Kazhdan's proof of the finite generation of lattices:

(i) A locally compact group with Property (T) is compactly generated, and in particular a discrete group with Property (T) is finitely generated.
(ii) For a local field $\mathbf{K}$ (in particular for $\mathbf{K} = \mathbf{R}$), the groups $SL_n(\mathbf{K})$, $n \geq 3$, and $Sp_{2n}(\mathbf{K})$, $n \geq 2$, have Property (T). This carries over to other groups $\mathbb{G}(\mathbf{K})$ of $\mathbf{K}$-rational points of appropriate simple algebraic groups, and in particular to simple real Lie groups of real rank at least two.

(iii) A lattice Γ in a locally compact group G has Property (T) if and only if G has it.

Chapter 2 concentrates on a property which was shown in the late 1970s to be equivalent to Kazhdan's property for a large class of groups:

> A topological group G has *Property (FH)* if any continuous action of G by affine isometries on a Hilbert space has a fixed point.

We have kept the discussions in Chapters 1 and 2 mostly independent of each other, so that either can be chosen as an introduction to our subject.

If π is a representation of a group G, let $H^1(G,\pi)$ denote the first cohomology space of π. It is straightforward to translate Property (FH) as a vanishing property: $H^1(G,\pi)=0$ for all unitary representations π of G. There are strong consequences on several types of actions: for a group with Property (FH), any isometric action on a tree has a fixed point or a fixed edge (this is *Property (FA)* of Serre), any isometric action on a real or complex hyperbolic space has a fixed point, and any action on the circle which is orientation preserving and smooth enough factors through a finite cyclic group (a result of Navas and Reznikov). There is also a reformulation of Property (FH) in terms of scalar-valued functions on the group: any function which is conditionally of negative type is bounded.

In the last section, we prove the Delorme–Guichardet theorem: for σ-compact locally compact groups, Properties (T) and (FH) are equivalent.

Chapter 3 is devoted to reduced cohomology spaces $\overline{H^1}(G,\pi)$, which are the Hausdorff spaces associated to the cohomology spaces $H^1(G,\pi)$ for the appropriate topology; $\overline{H^1}$ is from several points of view a "better" functor than H^1. For a compactly generated locally compact group G, Shalom has shown that several vanishing properties are equivalent, including:

(i) Property (T) or (FH), namely the vanishing of the cohomology space $H^1(G,\pi)$ for every unitary representation π of G;
(ii) the vanishing of the *reduced* cohomology space $\overline{H^1}(G,\pi)=0$ for every unitary representation π.

Shalom's result implies that a countable group with Property (T) is always the quotient of a finitely presented group with Property (T). This answers a natural question, since Property (T) implies finite generation (Kazhdan's observation) but not finite presentation (as shown by examples discovered later).

There is also a section on Kostant's result according to which the isometry group $Sp(n,1)$ of a quaternionic hyperbolic space ($n\geq 2$) has Property (T). The method of Section 3.3 uses properties of harmonic mappings and rests on ideas of Gromov.

To find examples of non-compact groups with Property (T), the only methods known from the time of Kazhdan's paper until about 30 years later have been to use at some point the theory of Lie groups or of algebraic groups, and the resulting fact that groups like $SL_n(\mathbf{K})$, $n \geq 3$, and $Sp_{2n}(\mathbf{K})$, $n \geq 2$, have Property (T). Chapter 4 focuses on another method, due to Shalom, that shows again Property (T) for $SL_n(\mathbf{Z})$, and as a bonus shows it also for $SL_n(R)$ for other rings R ($n \geq 3$).

The new notion entering the scene is that of bounded generation,[1] of which the relevance for Property (T) was pointed out by Colin de Verdière and Shalom. On the one hand, the method can be used to estimate various *Kazhdan constants*, namely to obtain quantitative sharpenings of the qualitative notion of Property (T). On the other hand, the groups to which the method applies need not be locally compact. For example, for $n \geq 3$, the loop group $LSL_n(\mathbf{C})$ consisting of all continuous functions from the circle to $SL_n(\mathbf{C})$ has Property (T).

Chapter 5 is an account of the so-called *spectral criterion*. More precisely, given a group Γ generated by a finite set S, there is a finite graph $\mathcal{G}(S)$ attached to the situation; if this graph is connected and if its smallest non-zero eigenvalue is strictly larger than $1/2$, then Γ has Property (T); moreover, the method gives an estimate of the Kazhdan constant for S. The spectral criterion is due to Zuk (1996) and Ballmann-Swiatkowski (1997); it relies on some fundamental work by Garland and Borel (1973), and is strongly used in more recent work of Gromov (2003) and others concerning random groups.

Chapter 6 is a small sample of applications of Property (T). We indicate a construction, due to Margulis, of finite graphs with good expanding properties. Then we discuss some applications to ergodic theory: estimates of spectral gaps for operators associated to appropriate actions, the importance for the so-called strongly ergodic actions (Schmidt and Connes–Weiss), and invariance of Property (T) by "measure equivalence" (work of Furman and Popa). The final section of Chapter 6 is about the Banach–Ruziewicz problem, which asks whether the normalised Lebesgue measure on the unit sphere $\mathbf{S}^{n-1}$ of $\mathbf{R}^n$ is the unique rotation-invariant *finitely additive* measure defined on all Lebesgue-measurable sets; the answer, which is positive for all $n \geq 3$, follows when $n \geq 5$ from the fact that the special orthogonal group $SO(n)$ contains a dense subgroup which has Property (T).

[1] A group Γ is boundedly generated if there exist a finite family $C_1, \ldots, C_k$ of cyclic subgroups in Γ and an integer $N \geq 1$ such that any $\gamma \in \Gamma$ is a product of at most N elements from the union $\cup_{1 \leq j \leq k} C_j$.

Despite their importance, applications to operator algebras in general and the work of Popa in particular are almost not discussed within this book.

Chapter 7 is a short collection of open problems which, at the time of writing, are standard in the subject.

A significant part of the theory of Property (T) uses the theory of unitary representations in a non-technical way. Accordingly, we use freely in this book the "soft parts" of representation theory, with as little formalism as possible. The reader who wishes to rely now and then on a more systematic exposition will find one in the appendix which appears as the second half of this book.

We discuss there some of the basic notions: generalities about unitary representations, invariant measures on homogeneous spaces, functions of positive type and functions conditionally of negative type, unitary representations of abelian groups, unitary induction, and weak containment. Moreover, we have one chapter on amenable groups, a notion which goes back to the time (1929) when von Neumann wrote up his view of the Hausdorff–Banach–Tarski paradox (itself from the period 1914–1924); amenability and Property (T) are two fundamental properties in our subject, and the second cannot be fully appreciated without some understanding of the first.

The size of the present book has grown to proportions that we did not plan! There are several much shorter introductions to the subject which can be recommended: the original Bourbaki seminar [DelKi–68], Chapter 7 of [Zimm–84a], Chapter III of [Margu–91], Chapter 3 of [Lubot–94], Chapter 5 of [Spatz–95], the Bourbaki seminar by one of us [Valet–04], and a book in preparation [LubZu] on *Property* (τ), which is a variant of Property (T). Despite its length, the present book is far from complete; the list of references should help the reader to appreciate the amount of material that we do not discuss.

Historical introduction

For readers who already have some knowledge of Property (T), here is our personal view on the history of this notion.

The first 25 years

First appearance of Property (T)

The subject of this book began precisely with a three page paper [Kazhd–67]. Kazhdan's insight was the key to many unexpected discoveries. Indeed, in the mathematical literature, there are very few papers with such a rich offspring. Property (T) is now a basic notion in domains as diverse as group theory, differential geometry, ergodic theory, potential theory, operator algebras, combinatorics, computer science, and the theory of algorithms.

On the one hand, Kazhdan defines a locally compact group G to have *Property (T)*, now also called the *Kazhdan Property*, if the unit representation[2] is isolated in the appropriate space of unitary representations of G. It is straightforward to show that a group G with this property is compactly generated and that its largest Hausdorff abelian quotient $G/\overline{[G,G]}$ is compact; in particular, a countable group Γ with Property (T) is finitely generated and its first homology group $H_1(\Gamma, \mathbf{Z}) = \Gamma/[\Gamma, \Gamma]$ is finite. On the other hand, Kazhdan shows that, besides compact groups, groups having Property (T) include $SL_n(\mathbf{K})$, $n \geq 3$, and $Sp_{2n}(\mathbf{K})$, $n \geq 2$, for any local field $\mathbf{K}$. This implies in particular that a simple real Lie group G with large real rank and with finite centre has Property (T); in Kazhdan's paper, "large" real rank l means $l \geq 3$, but shortly afterwards[3] it was checked that $l \geq 2$ is sufficient. Moreover, a lattice Γ in a locally compact group G has Property (T) if and only if G has Property (T).

One spectacular consequence of these results and observations can be phrased as follows. Let $M = \Gamma \setminus G/K$ be a locally symmetric Riemannian manifold of finite volume, where G is a connected semisimple Lie group with finite centre, with all factors of real ranks at least 2, and where K is a maximal compact subgroup of G. Then:

(i) the fundamental group $\Gamma = \pi_1(M)$ is finitely generated;
(ii) the first Betti number $b_1(M) = \dim_{\mathbf{R}} \mathrm{Hom}(\Gamma, \mathbf{R})$ is zero.

Statement (i) "gives a positive answer to part of a hypothesis of Siegel on the finiteness of the number of sides of a fundamental polygon" (the quotation is from [Kazhd–67]). Here is what *we* understand by the "hypothesis of Siegel": there exists a convenient fundamental domain for the action of Γ on G, namely a Borel subset $\Omega \subset G$ such that $(\gamma\Omega)_{\gamma \in \Gamma}$ is a partition of G, such that each element of G has a neighbourhood contained in a finite union of translates $\gamma\Omega$, and more importantly such that the set $S = \{\gamma \in \Gamma : \gamma\overline{\Omega} \cap \overline{\Omega} \neq \emptyset\}$ is finite; it follows that S generates Γ (Section 9 in [Siege–43]).

Before Kazhdan's paper, some results of vanishing cohomology had been obtained in [CalVe–60] and [Matsu–62]. Soon after Kazhdan's paper, it was also established[4] that any lattice Γ in a semisimple Lie group G is finitely

[2] The unit representation is also called the *trivial representation*, and "T" holds for "trivial".
[3] Indeed, only $SL_n(\mathbf{K})$ appears in Kazhdan's paper. Similar considerations hold for $Sp_{2n}(\mathbf{K})$, as was shown independently by [DelKi–68], [Vaser–68], and [Wang–69].
[4] Existence of nice fundamental domains for lattices was shown separately in the real rank one case [GarRa–70] and in the higher rank case [Margu–69]. See Chapter XIII in [Raghu–72], and an appendix of Margulis to the Russian translation of this book (the appendix has also appeared in English [Margu–84]).

generated, without any restriction on the ranks of the factors of G, and moreover the "Selberg conjecture" was proven.[5]

When M is not compact, it is often difficult to establish (i), and more generally finite generation for lattices in semisimple algebraic groups over local fields, by any other method than Property (T).[6]

Note however that some simple Lie groups of real rank one, more precisely the groups locally isomorphic to $SO(n,1)$ and $SU(n,1)$, do *not* have Property (T). Let us try and reconstruct the way this fact was realised.

For $SO(n,1)$, spherical functions of positive type have been determined independently by Vilenkin (see [Vilen–68]) and Takahashi [Takah–63]. (Particular cases have been known earlier: [Bargm–47] for $SL_2(\mathbf{R})$, [GelNa–47] and [Haris–47] for $SL_2(\mathbf{C})$.) As a consequence, it is clear that the unit representation is not isolated in the unitary dual of $SO(n,1)$, even if this is not explicitly stated by Takahashi and Vilenkin. In 1969, S.P. Wang writes in our terms that $SO(n,1)$ does not have the Kazhdan Property (Theorem 4.9 in [Wang–69]).

As far as we know, it is Kostant [Kosta–69] who first worked out the spherical irreducible representations of *all* simple Lie groups of real rank one (see below), and in particular who has first shown that $SU(n,1)$ does not have Property (T). This can be found again in several later publications, among which we would like to mention [FarHa–74].

For the related problem to find representations π of $G = SO(n,1)$ or $G = SU(n,1)$ with non-vanishing cohomology $H^1(G,\pi)$, see [VeGeG–73], [VeGeG–74], [Delor–77], and [Guic–77b]. The interest in non-vanishing spaces $H^1(G,\pi)$, and more generally $H^j(G,\pi)$ for $j \geq 1$, comes also from the following decomposition. Let G be a Lie group with finitely many connected

[5] Kazhdan and Margulis have shown that, if G is a connected linear semisimple Lie group without compact factor, there exists a neighbourhood W of e in G such that any discrete subgroup Γ in G has a conjugate $g\Gamma g^{-1}$ disjoint from $W \setminus \{e\}$. It follows that the volume of G/Γ is bounded below by that of W. By ingenious arguments, it also follows that, if Γ is moreover a lattice such that G/Γ is *not* compact, then Γ contains unipotent elements distinct from e (Selberg conjecture). See [KazMa–68], [Bore–69a], and Chapter XI in [Raghu–72].

[6] For lattices in real Lie groups, there is a proof by Gromov using the "Margulis Lemma" and Morse theory [BaGrS–85]; see [Gelan–04] for a simple account. Of course, it is classical that arithmetic lattices are finitely generated, indeed finitely presented (Theorem 6.12 in [BorHa–62]), and Margulis has shown that lattices in semisimple Lie groups of rank at least two are arithmetic; but Margulis' proof uses finite generation from the start. For lattices in algebraic groups defined over fields of characteristic zero, there is an approach to superrigidity and arithmeticity of lattices which does not use finite generation [Venka–93]; but this is not available in finite characteristic. Indeed, in characteristic p, lattices in rank one groups need not be finitely generated: see §II.2.5 in [Serre–77], as well as [Lubot–89] and [Lubot–91]. (However, still in characteristic p, irreducible lattices in products of at least two rank one groups are always finitely generated [Raghu–89].) To sum up, the use of the Kazhdan Property to prove finite generation of lattices is very efficient in most cases, and is currently unavoidable in some cases.

components and let K be a maximal compact subgroup; the quotient G/K is contractible, indeed homeomorphic to a Euclidean space (the Cartan–Iwasawa–Malcev–Mostow theorem). Let Γ be a torsion free cocompact lattice in G, so that $M = \Gamma \setminus G/K$ is both a closed manifold and an Eilenberg–McLane space $K(\Gamma, 1)$. There are integers $m(\Gamma, \pi) \geq 0$ such that

$$H^*(\Gamma, \mathbf{C}) = H^*(M, \mathbf{C}) = \bigoplus_{\pi \in \hat{G}} m(\Gamma, \pi) H^*(G, \pi)$$

where the summation can be restricted to those representations π in the unitary dual $\hat{G}$ of G such that $H^*(G, \pi) \neq 0$. See [BorWa–80], Chapters VI and VII.

Property (T) for the groups $Sp(n, 1)$ and $F_{4(-20)}$

Shortly after Kazhdan's paper, Kostant made an analysis of the spherical irreducible representations of simple Lie groups; his results were announced in [Kosta–69] and the detailed paper was published later [Kosta–75]. His aim was to demonstrate the irreducibility of a (not necessarily unitary) representation of G which is either in the so-called principal series, or in the complementary series (when the latter exists). Only in the very last line (of both the announcement and the detailed paper), Kostant relates his work to that of Kazhdan, establishing[7] that the rank one groups $Sp(n, 1)$, $n \geq 2$, and the rank one real form $F_{4(-20)}$ of the simple complex Lie group of type F_4, have Property (T), thus completing the hard work for the classification of simple real Lie groups with Property (T). As a consequence, the semisimple Lie groups having Property (T) are precisely the Lie groups[8] locally isomorphic to products of simple Lie groups with Lie algebras *not* of type $\mathfrak{so}(n, 1)$ or $\mathfrak{su}(n, 1)$.

Let $\mathbb{G}$ be a connected linear algebraic group defined over a local field $\mathbf{K}$; set $G = \mathbb{G}(\mathbf{K})$. For $\mathbb{G}$ semisimple and $\mathbf{K}$ a non-archimedean field, the situation is much simpler than if $\mathbf{K}$ is $\mathbf{R}$ or $\mathbf{C}$, since then G has Property (T) if and

[7] Now, we know several proofs that the groups $Sp(n, 1)$, $n \geq 2$, and $F_{4(-20)}$ have Property (T): the "cohomological proof" of Borel and Wallach (see Corollary 5.3 of Chapter V in [BorWa–80], and [HarVa–89]), a proof using harmonic analysis on groups of Heisenberg type (see [CowHa–89], the indication in [Cowli–90], and Theorem 1.16 in [Valet–94]), as well as proofs of Gromov, Korevaar-Schoen, Mok, Pansu, and others using Bochner's formula of differential geometry and properties of harmonic mappings defined on Riemannian symmetric spaces (see [KorSc–93], [Mok–95], [Pansu–95], [Pansu–98], and [Gromo–03, Item 3.7.D']). The last proofs are part of a theory of "geometric superrigidity", of which the first goal had been to put into a differential geometric setting the superrigidity theorem of Margulis; see, among others, [Corle–92], [GroSc–92], and [MoSiY–93].

[8] Strictly speaking, this was only clear in 1969 for groups with finite centre. It holds in the general case by Lemma 1.7 in [Wang–82], or by results of Serre first published in Sections 2.c and 3.d of [HarVa–89]. Serre's results on Property (T) for central extensions follow also simply from results of Shalom, as in §3.3 of [Valet–04].

only if it has no simple factor of **K**-rank one.[9] In the general case, S.P. Wang has determined when G has Property (T) in terms of a Levi decomposition of G (assuming that such a decomposition exists, this being always the case in characteristic zero); see [Wang–82], as well as [Shal–99b] and [Cornu–06d, Corollary 3.2.6].

Construction of expanding graphs and Property (T) for pairs

The first application of the Kazhdan Property outside group theory was the explicit construction by Margulis of remarkable families of finite graphs. In particular, for any degree $k \geq 3$, there are constructions of families of finite k-regular graphs which are *expanders*; this means that there exists a so-called *isoperimetric constant* $\varepsilon > 0$ such that, in each graph of the family, any non-empty subset A of the set V of vertices is connected to the complementary set by at least $\varepsilon \min\{\#A, \#(V \setminus A)\}$ edges. While the existence of such graphs is easily established on probabilistic grounds, explicit constructions require other methods.

A basic idea of [Margu–73] is that, if an infinite group Γ generated by a finite set S has Property (T) and is residually finite, then the finite quotients of Γ have Cayley graphs with respect to S which provide a family of the desired kind. Margulis' construction is explicit for the graphs, but does not provide explicit estimates for the isoperimetric constants. Constructions given together with lower bounds for these constants were given later, for example in [GabGa–81]; see also the discussion below on Kazhdan constants.[10]

Rather than Property (T) for one group, Margulis used there a formulation for a pair consisting of a group and a subgroup. This *Property (T) for pairs*, also called *relative Property (T)*, was already important in Kazhdan's paper, even though a name for it appears only in [Margu–82]. It has since become a basic notion, among other reasons for its role in operator algebras, as recognized by Popa. Recent progress involves defining Property (T) for a pair consisting of a group and a subset [Cornu–06d].

There is more on Property (T) for pairs and for semidirect products in [Ferno–06], [Shal–99b], [Valet–94], and [Valet–05].

[9] If $\mathbb{G}$ is connected, simple, and of **K**-rank one, then G acts properly on its Bruhat–Tits building [BruTi–72], which is a tree, and it follows that G does not have Property (T).

[10] More recently, there has been important work on finding more expanding families of graphs, sometimes with optimal or almost optimal constants. We wish to mention the so-called Ramanujan graphs, first constructed by Lubotzky–Phillips–Sarnak and Margulis (see the expositions of [Valet–97] and [DaSaV–03]), results of J. Friedman [Fried–91] based on random techniques, and the zig-zag construction of [ReVaW–02], [AlLuW–01]. Most of these constructions are related to some weak form of Property (T).

Group cohomology, affine isometric actions, and Property (FH)

Kazhdan's approach to Property (T) was expressed in terms of weak containment of unitary representations. There is an alternative approach involving group cohomology and affine isometric actions.

In the 1970s, cohomology of groups was a very active subject with (among many others) an influential paper by Serre. In particular (Item 2.3 in [Serre–71]), he conjectured that $H^i(\Gamma, \mathbf{R}) = 0$, $i \in \{1, \ldots, l-1\}$, for a cocompact discrete subgroup Γ in an appropriate linear algebraic group G "of rank l". This conjecture was partially solved by Garland in an important paper [Garla–73] which will again play a role in the later history of Property (T). Shortly after, S.P. Wang [Wang–74] showed that $H^1(G, \pi) = 0$ for a separable locally compact group G with Property (T), where π indicates here that the coefficient module of the cohomology is a finite-dimensional Hilbert space on which G acts by a unitary representation.

In 1977, Delorme showed that, for a topological group G, Property (T) implies that $H^1(G, \pi) = 0$ for all unitary representations π of G [Delor–77]. Previously, for a group G which is locally compact and σ-compact, Guichardet had shown that the converse holds (see [Guic–72a], even if the expression "Property (T)" does not appear there, and [Guic–77a]). Delorme's motivation to study 1-cohomology was the construction of unitary representations by continuous tensor products.

A topological group G is said to have *Property (FH)* if every continuous action of G by affine isometries on a Hilbert space has a fixed point. It is straightforward to check that this property is equivalent to the vanishing of $H^1(G, \pi)$ for all unitary representations π of G, but this formulation was not standard before Serre used it in talks (unpublished). Today, we formulate the result of Delorme and Guichardet like this: a σ-compact[11] locally compact group has Property (T) if and only if it has Property (FH).

Recall that there is a *Property (FA)* for groups, the property of having fixed points for all actions by automorphisms on trees. It was first studied in [Serre–74] (see also §I.6 in [Serre–77]); it is implied by Property (FH) [Watat–82].

We make one more remark, in order to resist the temptation of oversimplifying history. Delorme and Guichardet also showed that a σ-compact locally compact group G has Property (T) if and only if all real-valued continuous

[11] The hypothesis of σ-compactness is necessary, since there exist discrete groups with Property (FH) which are not countable, and therefore which are not finitely generated, and even less with Property (T). An example is the group of all permutations of an infinite set, with the discrete topology [Cornu–06c].

functions on G which are conditionally of negative type are bounded.[12] On the one hand, this was independently rediscovered and proved by other methods in [AkeWa–81], a paper in which the authors seek to understand the unitary dual of a group in terms of functions of conditionally negative type, and also a paper written under the strong influence of Haagerup's work on reduced C*-algebras of non-abelian free groups [Haage–78]. On the other hand, a weak form of the same result on Property (T) and functions of negative type appeared earlier in [FarHa–74], itself motivated by the wish to understand which are the invariant metrics on a space of the form G/K which are induced by embeddings in Hilbert spaces.

Whatever the early history has been, there has been a growing interest in the point of view of Property (FH) which is now considered as basic. Recently, it has been shown that, for locally compact groups, Property (T) implies the property of having fixed points for affine isometric actions on various Banach spaces, for example[13] on spaces of the form $L^p(\mu)$ with $1 < p \leq 2$ [BaFGM].

Normal subgroups in lattices

Let G be a connected semisimple Lie group with finite centre, without compact factor, and of real rank at least 2. Let Γ be an irreducible lattice in G and let N be a normal subgroup of Γ. Margulis has shown that

either N is of finite index in Γ;
or N is central in G, and in particular finite.

Property (T) is a crucial ingredient of Margulis' proof. Indeed, for non-central N, the proof of the finiteness of the quotient cannot rely on any size estimate since $\#(\Gamma/N)$ can be arbitrarily large (think of congruence subgroups in $SL_3(\mathbf{Z})$). The strategy is to show that Γ/N is amenable and has Property (T), and it follows that Γ/N is a finite group.[14] (In the special case where all factors of G have real ranks at least 2, Property (T) for Γ/N is straightforward.)

[12] This reformulation is particularly well suited to Coxeter groups. Let (W, S) be a Coxeter system. The S-word length $\ell_S : W \longrightarrow \mathbf{R}_+$ is conditionally of negative type [BoJaS–88]. It follows that W cannot have Property (T) as soon as W is infinite.

[13] This cannot hold for a Banach space of the form $C(X)$; see Remark 8.c of Chapter 4 in [HarVa–89]. More on this in Exercises 1.8.20 and 2.14.12.

[14] Though stated here for real semisimple Lie groups, this result holds for G of rank at least 2 in a much larger setting. See [Margu–78] and [Margu–79], as well as Theorems IV.4.9 (page 167) and IX.5.6 (page 325) in [Margu–91]. The Normal Subgroup Theorem and the strategy for its proof have been extended to irreducible lattices in products $\mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ of automorphism groups of trees [BurMo–00] and in a larger family of products of two locally compact groups. One of the consequences is that the result holds for many Kac–Moody groups [BadSh–06].