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PARTIAL DIFFERENTIAL EQUATIONS FOR PROBABILISTS

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This book is dedicated to the memory of my friend Eugene Fabes.

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Preface

There are few benefits to growing old, especially if you are a mathematician. However, one of them is that, over the course of time, you accumulate a certain amount of baggage containing information in which, if you are lucky and they are polite, your younger colleagues may express some interest.

Having spent most of my career at the interface between probability and partial differential equations, it is hardly surprising that this is the item in my baggage about which I am asked most often. When I was a student, probabilists were still smitten by the abstract theory of Markov processes which grew out of the beautiful work of G. Hunt, E.B. Dynkin, R.M. Blumenthal, R.K. Getoor, P.A. Meyer, and a host of others. However, as time passed, it became increasingly apparent that the abstract theory would languish if it were not fed a steady diet of hard, analytic facts. As A.N. Kolmogorov showed a long time ago, ultimately partial differential equations are the engine which drives the machinery of Markov processes. Until you solve those equations, the abstract theory remains a collection of "if, then" statements waiting for someone to verify that they are not vacuous.

Unfortunately for probabilists, the verification usually involves ideas and techniques which they find unpalatable. The strength of probability theory is that it deals with probability measures, but this is also its weakness. Because they model a concrete idea, probability measures have enormous intuitive appeal, much greater than that of functions. They are particularly useful when comparing the relative size of quantities: A is larger than B because it is more likely. For example, it is completely obvious that a diffusion is less likely to go from x to y before leaving a set Γ than it is to go from x to y when it is allowed to leave Γ. On the other hand, probability theory provides little help when it comes to determining whether one can talk sensibly about "the probability of going from x to y," with or without leaving Γ. Indeed, in a continuous setting, such events usually will have probability 0, and so one needs to discuss their probabilities in terms of (preferably smooth) densities with respect to a reference measure like Lebesgue's. Proving that such a density exists, much less checking that it possesses desirable properties, is not something for which probability reasoning is particularly well suited. As a consequence, probabilists have tended to avoid addressing these questions themselves and have relied on the hard work of analysts.

My purpose in writing this book has been to provide probabilists with a few tools with which they can understand how to prove on their own some of the basic facts about the partial differential equations with which they deal. In so far as possible, I have tried to base my treatment on ideas which are already familiar to anyone who has worked in stochastic analysis. In fact, there is nothing in the first two chapters which requires more than a course on measure theory (including a little Fourier analysis) and a semester of graduate-level probability theory. In Chapter 1, I prove a general existence result for solutions to Kolmogorov's forward equation by a method which is essentially due to K. Itô, even though it may take all but my more sophisticated readers a little time to recognize it as such. In summary, the upshot of Chapter 1 is that solutions nearly always exist, at least as probability measures. Chapter 2 is an initial attempt to prove that the solutions produced in Chapter 1 can be used to generate smooth solutions to Kolmogorov's backward equation, at least when the initial data are themselves smooth. Again, the ideas here will be familiar to anyone who has worked with stochastic integral equations. In fact, after seeing some of the contortions which I have to make for not doing so, experts will undoubtedly feel that I have paid a high price for not using Brownian motion. Be that as it may, the conclusion drawn in Chapter 2 is that solutions to Kolmogorov's backward equation preserve the regularity properties their initial data possess.

All the results in Chapters 1 and 2 can be viewed as translations to a measure setting of results which are well known for flows generated by a vector field. It is not until Chapter 3 that I begin discussing properties which are not shared by deterministic flows. Namely, when the diffusion coefficients in Kolmogorov's equation are "elliptic" (i.e., the diffusion matrix is strictly positive definite), the associated flow not only preserves but even smooths the initial data. The classic example of this smoothing property is the standard heat equation which immediately transforms any reasonably bounded initial data into a smooth (in fact, analytic) function. Until quite recently, probabilists have been at a complete loss when it came to proving such results in any generality. However, thanks to P. Malliavin, we now know a method which allows us to transfer via integration by parts smoothness properties of the Gauss distribution to the measures produced by Itô's construction. Malliavin himself and his disciples, like me, implemented his ideas in a pathspace context. However, if one is satisfied with less than optimal conclusions, then there is no need to work in pathspace, and there are good reasons not to. Specifically, the price one pays for the awkward treatment of Itô's theory in Chapters 1 and 2 turns out to buy one a tremendous technical advantage in Chapter 3. Briefly stated, the advantage is that solutions to Itô's stochastic differential equations are not classically smooth functions of the driving Brownian motion, even though they are infinitely smooth in the sense of Sobolev. Thus, to carry out Malliavin's program in pathspace, one has to jump through all sorts of annoying technical hoops. On the other hand, if you do all your integration by parts before passing to the limit in Itô's procedure, you avoid all those difficulties, and that is the reason why I adopted the approach which I did in Chapters 1 and 2.

By the end of Chapter 3, I have derived the basic regularity properties for solutions to Kolmogorov's equations with smooth coefficients which are uniformly elliptic. In particular, I have shown that the transition probability function admits a smooth density which, together with its derivatives, satisfies Gaussian estimates, and I have used these to derive Weyl's Lemma (i.e., hypoellipticity) for solutions to the associated elliptic and parabolic equations. For many applications to probability theory, this is all that one needs to know. However, for other applications, it is important to have more refined results, and perhaps the most crucial such refinement is the estimation of the transition probability density from below. In Chapter 4, I develop quite sharp upper and lower bounds on the transition probability using a methodology which has essentially nothing to do with probability theory. Instead of probability theory, the origin of the ideas here come from the calculus of variations, and so the usual form of Kolmogorov's equations gets replaced by their divergence form counterparts. As long as the coefficients are sufficiently smooth, this replacement hardly affects the generality of the conclusions derived. However, it has enormous impact on the mathematics used to draw those conclusions. In essence, everything derived in the first three chapters relies on the minimum principle (i.e., non-negativity preservation). By writing the equation in divergence form, a second powerful mathematical tool is made manifest: the theory of self-adjoint operators and their spectral theory. In his brilliant article about parabolic equations, J. Nash showed how, in conjunction with the minimum principle, self-adjointness can be used to prove surprising estimates for solutions of parabolic equations written in divergence form. Not only are these estimates remarkably sharp, they make no demands on the regularity of the coefficients involved. As a result, they are entirely different from the results derived in Chapter 3, all of which rely heavily on the smoothness of the coefficients. Supplementing Nash's ideas with a few more recent ones, I derive in Chapter 4 very tight upper and lower bounds on the transition probability density. The techniques used are all quite elementary, but the details are intricate.

In Chapter 5, I return to probabilistic techniques to localize the results proved earlier. Here the demands on the reader's probabilistic background are much greater than those made earlier. In particular, the reader is assumed to know how to pass from a transition probability function to a Markov process on pathspace. I have summarized some of the ideas involved, but I doubt if my summary will be sufficient for the uninitiated. In any case, for those who have the requisite background, Chapter 5 not only provides a ubiquitous localization procedure but also a couple of important applications of the results to which it leads. In particular, the chapter closes with proofs of both Nash's Continuity Theorem and Moser's extension of De Giorgi's Harnack principle. Chapter 6 can be viewed as a further application of the localization procedure developed in Chapter 5. Now the goal is to work on a differentiable manifold and to show how one can lift everything to that venue. Besides a familiarity with probability theory, the reader of Chapter 6 is assumed to have some acquaintance with the basic ideas of Riemannian differential geometry.

The concluding chapter, Chapter 7, represents an abrupt departure from the ones preceding it. Whereas in Chapter 4 the minimum principle still plays a role, in Chapter 7 it completely disappears. All the techniques introduced in Chapter 7 derive from Fourier analysis. I begin with a brief resumé of basic facts about Sobolev spaces. This is followed by a short course on pseudodifferential operators, one in which I avoid all but the most essential ingredients. I then apply pseudodifferential operators to prove far-reaching extensions of Weyl's Lemma, first for general, scalar valued, elliptic operators and then, following J.J. Kohn, for an interesting class of second order, degenerate operators with real valued coefficients. The latter extension, which is due to L. Hörmander, is the one of greater interest to probabilists. Indeed, it has already played a major role in many applications and promises to continue doing so in the future.

Even though this book covers much of the material about partial differential equations needed by probabilists, it does not cover it all. Perhaps the most egregious omission is the powerful ideas introduced by M. Crandell and P.L. Lions under the name of "viscosity solutions." In a very precise sense, their theory allows one to describe a "probabilistic solution" without reference to probability theory. The advantage to this is that it removes the need to develop the whole pathspace apparatus in order to get at one's solution, an advantage that becomes decisive when dealing with non-linear equations like those which arise in optimal control and free boundary value problems. For the reader who wishes to learn about this important topic, I know of no better place to begin than the superb book [14] by L.C. Evans. Less serious is my decision to deal only with time independent coefficients. For the material in Chapters 1 and 2 it makes no difference, since as long as one is making no use of ellipticity, one can introduce time as a new coordinate. However, it does make some difference in the later chapters, but not enough difference to persuade me to burden the whole text with the additional notation and considerations which the inclusion of time dependence would have required.

Each chapter ends with a section entitled "Historical Notes and Commentary." The reader should approach these sections with a healthy level of skepticism. I have not spent much time tracking down sufficient historical evidence to make me confident that I have always given credit where credit is due and withheld it where it is not due. Thus, these sections should be read for what they are: a highly prejudiced, impressionistic account of what I think may be the truth.

Finally, a word about Eugene Fabes, to whom I have dedicated this book. Gene and I met as competitors, he working with Nestor Riviere to develop the analytic theory of parabolic equations with continuous coefficients and I working with S.R.S. Varadhan to develop the corresponding probabilistic theory. However, as anyone who knew him knows, Gene was not someone with whom you could maintain an adversarial relationship for long. After spending a semester together in Minnesota talking mathematics, sharing smoked fish, and drinking martinis, we became fast friends and eventually collaborated on two articles. To my great sorrow, Gene died too young. Just how much too young becomes increasingly clear with each passing year.

Daniel W. Stroock