

Index

- $O(n, \mathbf{R})$, 9
- $O^+(2, 1)$, 109
- $PGL(2, \mathbf{C})$, 41
- $PSL(2, \mathbf{C})$, 41
- $PSL(2, \mathbf{R})$, 92, 100
 - finite subgroups, 114
 - transitive on H , 93
 - transitive on hyperbolic lines, 93
- $PSU(2)$, 42
- $SO(3)$, 10
 - finite subgroups, 32–34
- $SO(n)$, 9
- $SU(2)$, 42
 - finite subgroups, 45
- ϵ -neighbourhood, 55, 69

- antipodal points, 40, 47, 65
- Archimedes theorem, 120
- area
 - double lune, 35
 - embedded torus, 130
 - hyperbolic circle, 98
 - hyperbolic polygon, 104
 - hyperbolic triangle, 103
 - in Riemannian metric, 86
 - independent of parametrization, 120
 - integral, 159
 - on abstract surface, 156
 - on embedded surface, 119
 - preserved under isometry, 86
 - spherical circle, 46, 48
 - spherical polygon, 37, 48
 - spherical triangle, 34
- atlas, 117, 156
 - examples on sphere, 117
 - on torus, 118
- binary
 - icosahedral group, 45
 - octahedral group, 45
 - tetrahedral group, 45
- Bolzano–Weierstrass theorem, 16
- building blocks, 172

- Calculus of Variations, 133–138
- Cambridge, very flat, 75
- catenary, 137
- catenoid, 137, 175
- Cauchy sequence, 15
- Cauchy–Riemann equations, 77
- Cauchy–Schwarz inequality, 2, 140
- Chain Rule, 77, 78, 116, 118, 145
- chart, 117, 156
- circular cylinder, 120, 123, 125, 130, 170
 - geodesics, 142
- circular half-cone, 150, 176
- classical geometries, 156, 164
 - curvature, 159
 - geodesic polars, 148
- compact space, 15, 22, 23
 - closed subset, 16
 - continuous image of, 17
- complement of curve
 - bounded component, 22, 23
 - unbounded component, 22, 23
- complete, 15, 98, 151
 - geodesic, 151
- complex analytic functions, 76
 - preserve angles, 77
- concatenation of curves, 20
- conformal
 - map, 87
 - to Euclidean metric, 80
- congruent triangles
 - hyperbolic, 113
 - spherical, 48
- connected, 11, 155
 - path connected, 11, 22

- continuous
 - branch of argument, 19
 - branch of logarithm, 19
 - function, 3
 - map, 3
 - uniformly, 16
- convex
 - polygons, 67
 - spherical polygons, 36
 - strongly, 166
 - subset of torus, 52, 53, 67
 - subsets, 36, 166
- convexity, 166–167
 - of hyperbolic triangles, 102, 104, 113
 - strong, 166
- Cramer's Rule, 79
- cross-ratio, 42, 48
- curvature
 - compact embedded surface, 127
 - concentrated at points, 172
 - ellipsoid, 131
 - embedded hyperboloid, 131
 - Gaussian, 124, 158
 - geometric definition, 163, 164
 - of abstract surface, 158, 163
 - of embedded surface, 124, 153–155
 - of embedded torus, 130
 - of plane curve, 123
 - sphere, 129
 - surface of revolution, 128
- curve
 - closed, 17
 - concatenation of, 20
 - in metric space, 11
 - index of, 19
 - length, 12, 80, 92, 118
 - on sphere, 29–31
 - plane, 123
 - polygonal, 17, 53
 - regular, 121, 130
 - simple, 17
 - simple closed, 17, 18
 - polygonal, 18, 20, 36, 54, 71, 169
 - simple polygonal, 60, 71, 169
 - smoothly immersed, 83
 - winding number, 19–20, 72
- curved spaces, 176
- cuspidal cubic, 83
- cylindrical end, 170

- derivatives
 - continuous partial, 76
 - matrix of partial, 76
 - of maps $\mathbf{R}^n \rightarrow \mathbf{R}^m$, 76
- diameter
 - of subset, 58
 - of triangle, 113
- dicyclic group, 45
- diffeomorphism, 155
 - between open subsets of \mathbf{R}^n , 79, 85, 116
 - between surfaces, 157
 - local, 116
- differentials, 76–78
- dihedral group, 22, 32, 52, 67, 114, 130
- direct isometry
 - embedded torus, 130
 - Euclidean space, 9
 - hyperbolic plane, 101, 102
- dissection, 12, 29, 95
 - mesh, 12, 30
- distance
 - between subsets, 23, 73, 106
 - from subset, 23
 - hyperbolic, 94, 97
 - Riemannian, 83
 - spherical, 25
- dodecahedron, 33, 48
- double cover, 41
 - of $SO(3)$, 42
 - plate experiment, 45
 - of Klein bottle, 68
 - of real projective plane, 65
- double lune, 34
 - area of, 35

- embedded surface, 53, 115–131
- energy
 - of smooth curve, 118, 133
- Escher, M. C., 34
- Euclidean
 - inner-product, 1
 - norm, 1
 - polygon, 22
- Euclidean geometry, 1–2, 5–11, 14–15, 17–23
- Euclidean space, 1
 - direct isometry, 9
 - rigid motion, 5
- Euler characteristic, *see* Euler number
- Euler number
 - g -holed torus, 64, 172, 173
 - compact polyhedron, 174
 - convex polyhedron, 60
 - Klein bottle, 67, 68
 - of polygonal decomposition, 59
 - of triangulation, 56, 63
 - real projective plane, 66
 - topological invariance, 62, 165, 169
 - virtual, 171
- Euler–Lagrange equations, 135, 136

INDEX

183

- fundamental form
 - first, 118
 - second, 124, 125
- Gauss's lemma, 145
- Gauss's Theorema Egregium, 155
- Gauss–Bonnet theorem
 - discrete, 174
 - Euclidean, 34
 - Euclidean polygons, 55
 - geodesic polygons, 167
 - geodesic triangles, 159–164, 167, 175
 - global, 168
 - hyperbolic, 103
 - hyperbolic polygons, 104
 - on torus, 55, 175
 - spherical, 34
 - spherical polygons, 36–39
- Gaussian curvature, 124
 - independent of parametrization, 126
- genus, 62, 64
- geodesic, 137
 - circles, 145, 148, 150, 164
 - area, 164
 - circumference, 164
 - complete, 151
 - energy minimizing, 147
 - equations, 137
 - germ, 141
 - has constant speed, 140, 141
 - in hyperbolic plane, 137, 142
 - in metric space, 13
 - length minimizing, 147
 - line, 53–54, 65, 67
 - line segment, 53, 173
 - local existence, 141
 - locally energy minimizing, 139, 140
 - locally length minimizing, 141
 - on abstract surface, 156
 - on embedded surface, 138–140
 - on embedded torus, 144
 - on sphere, 142
 - on surface of revolution, 143
 - polar coordinates, 145, 147, 153, 155, 158
 - polygons, 54, 159, 167
 - rays, 145
 - segment, 53, 148, 160, 168
 - in normal neighbourhood, 149
 - space, 13, 31
 - triangle, 159
 - angles, 160, 161, 163
 - convexity, 167
- geodesic equations
 - for embedded surface, 139
 - surfaces of revolution, 143
- great circle, 25, 47
- Heine–Borel theorem, 16
- Hessian of function, 124
- Hilbert's theorem, 131, 156
- homeomorphism, 3, 115, 121
- homology groups, 56, 170, 176
- Hopf–Rinow theorem, 151
- hyperbolic
 - area, 102
 - circles, 97–98
 - cosine formula, 111
 - distance, 94, 97
 - length, 95
 - line, 93, 96, 137
 - line segment, 96
 - metric, 89–92
 - perpendicular bisector, 101
 - polygons, 104, 105
 - sine formula, 111
 - triangles, 102–104, 113
- hyperbolic plane, 89–114
 - disc model, 89–92, 95–98, 109
 - hyperboloid model, 107–111
 - upper half-plane model, 92–95
- hyperboloid, 107
 - embedded, 131
 - upper sheet, 107
- icosahedron, 33
- internal diagonal of polygon, 37
- intrinsic metric, 13, 31, 52, 83, 119, 157
 - induced, 13
- Inverse Function theorem, 79, 116, 145
- isometric embedding, 5
- isometry
 - between metric spaces, 4, 67
 - between Poincaré models, 96, 112, 158
 - between surfaces, 157
 - disc model, 96–97
 - hyperboloid model, 109–110
 - of Riemannian metrics, 85, 92, 118
 - preserves areas, 86, 157
 - preserves intrinsic metric, 157
 - preserves lengths, 86, 157
 - upper half-plane, 92
- isometry group
 - $\text{Isom}(S^2)$, 31
 - $\text{Isom}(X)$, 5
 - $\text{Isom}(\mathbf{R}^n)$, 5
 - disc model, 112
 - embedded torus, 130
 - metric space, 5
 - transitive action, 5, 67
 - upper half-plane, 101
- Jacobian matrix, 76, 85, 116, 117
- Jordan Curve theorem, 17

- Klein bottle, 67
- lattice, 52
 - unit square, 52
- length
 - of curve, 12, 29–31, 92, 95, 118
 - space, 13
- local isometry, 157
- locally convex vertex, 37, 104, 168
- locally Euclidean torus, 52, 157, 169, 174
- Lorentzian inner-product, 107
- mesh of dissection, 12, 30
- metric, 1, 2
 - British Rail, 3
 - intrinsic, 13, 31, 52, 83, 119, 157
 - locally Euclidean, 52, 64, 67, 68, 123, 158, 169, 172, 173
 - locally hyperbolic, 65, 105, 158
 - London Underground, 4, 13
- metric spaces, 2–5, 11–17
 - closed set, 3
 - compact, 15, 22, 23
 - complete, 15, 151
 - connected, 11
 - homeomorphism, 3
 - locally path connected, 12
 - open ball, 3
 - open neighbourhood, 3
 - open set, 3
 - path connected, 11, 22
 - sequentially compact, 16
- Minding's theorem, 164
- Möbius geometry, 39–48
- Möbius strip, 65
- Möbius transformations, 39–42
 - on unit disc, 96
 - preserve angles, 48
 - preserve circles/straight lines, 41
 - real coefficients, 92
 - triply transitive, 41
- moving frame, 153
- non-degenerate point, 125
- norm
 - Euclidean, 1
 - Riemannian, 119
- normal neighbourhood, 145, 149, 158
 - strong, 158, 160, 166
- octahedron, 33
- orthogonal
 - group, 9
 - matrix, 5
- pair of pants, 171
- parallel lines
 - Euclidean, 105
 - hyperbolic, 105
- parametrization
 - constant speed, 137, 140
 - monotonic, 31, 94, 95
 - smooth, 115, 130
 - unit speed, 83, 121, 128, 142
- path, 11
 - connected, 11
- piecewise continuously differentiable, 14, 15
- plumbing, 171
- Poincaré
 - conjecture, 62
 - disc model, 89–92, 95–98, 109
 - models of hyperbolic plane, 89–107
 - upper half-plane model, 92–95
- polar triangle, 29
- polygon
 - Euclidean, 22
 - geodesic, 54, 159, 167
 - hyperbolic, 104
 - on locally Euclidean torus, 54
 - spherical, 36
- polygonal
 - approximation, 61, 68–73, 169
 - decomposition, 59–62, 67–73, 165, 168, 169
 - edges, 59
 - Euler number, 59
 - faces, 59
- positive imaginary axis L^+ , 93
- pseudosphere, 131
- real projective plane, 65
- reflections
 - composite of, 8, 32, 102
 - in affine hyperplane, 7
 - in hyperbolic line, 98–102, 112
 - in spherical line, 32
 - rotated, 10
- regular curve, 121, 130
- Riemannian metric
 - conformally Euclidean, 80
 - defines intrinsic metric, 83–84, 157
 - disc model, 89
 - distance, 83
 - existence, 165
 - hyperbolic, 89–92
 - in geodesic polars, 147
 - initial asymptotics, 147
 - norm, 119
 - on abstract surface, 156
 - on open subset of \mathbf{R}^2 , 79–88, 133

INDEX

185

- scaling, 164
 - upper half-plane, 91
- rigid motion, 5
- rotated reflection, 10, 23
- rotations of regular solids, 32, 33

- saddle point, 125
- sectional curvatures, 176
- sequentially compact, 16, 22
- simplices, 56
- simply connected, 45
- smooth
 - abstract surface, 155
 - embedded surface, 115–131
 - parametrized, 115
 - unparametrized, 131
 - map, 76
 - between abstract surfaces, 157
 - parametrization, 115, 130
- smooth curve
 - action, 119
 - arc-length, 83, 141, 162
 - energy, 118, 133, 140, 156
 - length, 14, 80, 82–85, 118, 140, 156
 - on abstract surface, 156
 - proper variation, 134
 - speed, 80
 - variation, 134
- smoothly embedded curve, 121
- smoothly immersed curve, 83, 121
- special orthogonal group, 9
- spherical
 - circles, 45–48
 - cosine formula, 27
 - distance, 25
 - line, 25
 - metric, 28
 - polygons, 36–39
 - Pythagoras, 27
 - second cosine formula, 29
 - sine formula, 27
 - triangle inequality, 28
 - triangles, 26–29, 33–36, 47–48, 55, 67
- spherical defect of vertex, 174
- spherical geometry, 25–39, 45–49, 82
- stationary point for energy, 138
- stereographic projection, 39, 42, 81, 117
 - from hyperboloid, 107
- strong normal neighbourhood, 158, 160, 166
- strongly convex, 166
- surface
 - abstract smooth, 155
 - closed, 156, 168
 - embedded, 53
 - unit normal, 117, 126, 150
 - non-orientable, 65
 - open, 170
 - orientable, 65
 - smooth embedded, 115
 - topological, 65
- surfaces of revolution, 121–123
 - curvature, 128–130
 - first fundamental form, 122
 - geodesic equations, 143
 - geodesics, 143
 - meridians, 122, 143
 - minimal, 135, 136
 - parallels, 122, 143
- symmetry group
 - cube, 23, 34
 - dodecahedron, 34
 - of metric space, 5
 - tetrahedron, 11, 34

- tangent space
 - independent of parametrization, 116
 - to embedded surface, 115
 - to hyperboloid, 109
 - to sphere, 81
- Taylor's theorem
 - on embedded surface, 124, 127
- tessellation, 33, 48
- topological
 - equivalence, 3
 - manifold, 3
 - space, 11, 15, 63
 - triangle, 55, 67
 - triangulation, 56, 165, 169–170
- torus, 51–55
 - g -holed, 62–65, 105, 171
 - convex subset, 52, 53
 - distance function, 51
 - embedded in \mathbf{R}^3 , 52
 - fundamental square, 51
 - locally Euclidean, 52
 - rectangular, 172
- transition functions, 156
 - being isometries, 156
- transitive action, 5, 94, 96, 110
 - isometry group, 5, 67, 93
 - triply, 41
- triangle inequality, 1, 83
 - Euclidean, 2
 - spherical, 28
- triangulation
 - existence, 165
 - geodesic, 33, 57, 165
 - of Klein bottle, 68
 - of sphere, 56, 62

- of torus, 57, 62
- subdivision, 58–59, 168, 169
- topological, 56, 165, 169–170

- ultraparallel hyperbolic lines, 105, 113
- uniformly continuous, 16
- unit normal
 - to embedded surface, 117, 126, 150
 - to plane curve, 123

- upper half-plane, 90
 - metric, 91

- variation
 - of smooth curve, 134
 - proper, 134

- winding numbers, 19–20, 72