

Curved Spaces

This self-contained textbook presents an exposition of the well-known classical two-dimensional geometries, such as Euclidean, spherical, hyperbolic and the locally Euclidean torus, and introduces the basic concepts of Euler numbers for topological triangulations and Riemannian metrics. The careful discussion of these classical examples provides students with an introduction to the more general theory of curved spaces developed later in the book, as represented by embedded surfaces in Euclidean 3-space, and their generalization to abstract surfaces equipped with Riemannian metrics. Themes running throughout include those of geodesic curves, polygonal approximations to triangulations, Gaussian curvature, and the link to topology provided by the Gauss–Bonnet theorem.

Numerous diagrams help bring the key points to life and helpful examples and exercises are included to aid understanding. Throughout the emphasis is placed on explicit proofs, making this text ideal for any student with a basic background in analysis and algebra.

PELHAM WILSON is Professor of Algebraic Geometry in the Department of Pure Mathematics, University of Cambridge. He has been a Fellow of Trinity College since 1981 and has held visiting positions at universities and research institutes worldwide, including Kyoto University and the Max-Planck-Institute for Mathematics in Bonn. Professor Wilson has over 30 years of extensive experience of undergraduate teaching in mathematics, and his research interests include complex algebraic varieties, Calabi–Yau threefolds, mirror symmetry and special Lagrangian submanifolds.

Curved Spaces

From Classical Geometries to Elementary Differential Geometry

P. M. H. Wilson

Department of Pure Mathematics, University of Cambridge,
and Trinity College, Cambridge



Cambridge University Press
978-0-521-88629-1 — Curved Spaces
P. M. H. Wilson
Frontmatter
[More Information](#)

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo
Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK
Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521886291

© P. M. H. Wilson 2008

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2008

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-88629-1 hardback
ISBN 978-0-521-71390-0 paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

For Stanzi, Toby and Alexia,
in the hope that one day
they might understand what is written herein,
and to Sibylle.

Contents

<i>Preface</i>	<i>page ix</i>
1 Euclidean geometry	1
1.1 Euclidean space	1
1.2 Isometries	4
1.3 The group $O(3, \mathbf{R})$	9
1.4 Curves and their lengths	11
1.5 Completeness and compactness	15
1.6 Polygons in the Euclidean plane	17
<i>Exercises</i>	22
2 Spherical geometry	25
2.1 Introduction	25
2.2 Spherical triangles	26
2.3 Curves on the sphere	29
2.4 Finite groups of isometries	31
2.5 Gauss–Bonnet and spherical polygons	34
2.6 Möbius geometry	39
2.7 The double cover of $SO(3)$	42
2.8 Circles on S^2	45
<i>Exercises</i>	47
3 Triangulations and Euler numbers	51
3.1 Geometry of the torus	51
3.2 Triangulations	55
3.3 Polygonal decompositions	59
3.4 Topology of the g -holed torus	62
<i>Exercises</i>	67
Appendix on polygonal approximations	68
4 Riemannian metrics	75
4.1 Revision on derivatives and the Chain Rule	75
4.2 Riemannian metrics on open subsets of \mathbf{R}^2	79

4.3	Lengths of curves	82
4.4	Isometries and areas	85
	<i>Exercises</i>	87
5	Hyperbolic geometry	89
5.1	Poincaré models for the hyperbolic plane	89
5.2	Geometry of the upper half-plane model H	92
5.3	Geometry of the disc model D	96
5.4	Reflections in hyperbolic lines	98
5.5	Hyperbolic triangles	102
5.6	Parallel and ultraparallel lines	105
5.7	Hyperboloid model of the hyperbolic plane	107
	<i>Exercises</i>	112
6	Smooth embedded surfaces	115
6.1	Smooth parametrizations	115
6.2	Lengths and areas	118
6.3	Surfaces of revolution	121
6.4	Gaussian curvature of embedded surfaces	123
	<i>Exercises</i>	130
7	Geodesics	133
7.1	Variations of smooth curves	133
7.2	Geodesics on embedded surfaces	138
7.3	Length and energy	140
7.4	Existence of geodesics	141
7.5	Geodesic polars and Gauss's lemma	144
	<i>Exercises</i>	150
8	Abstract surfaces and Gauss–Bonnet	153
8.1	Gauss's Theorema Egregium	153
8.2	Abstract smooth surfaces and isometries	155
8.3	Gauss–Bonnet for geodesic triangles	159
8.4	Gauss–Bonnet for general closed surfaces	165
8.5	Plumbing joints and building blocks	170
	<i>Exercises</i>	175
	<i>Postscript</i>	177
	<i>References</i>	179
	<i>Index</i>	181

Preface

This book represents an expansion of the author's lecture notes for a course in Geometry, given in the second year of the Cambridge Mathematical Tripos. Geometry tends to be a neglected part of many undergraduate mathematics courses, despite the recent history of both mathematics and theoretical physics being marked by the continuing importance of geometrical ideas. When an undergraduate geometry course is given, it is often in a form which covers various assorted topics, without necessarily having an underlying theme or philosophy — the author has in the past given such courses himself. One of the aims in this volume has been to set the well-known classical two-dimensional geometries, Euclidean, spherical and hyperbolic, in a more general context, so that certain geometrical themes run throughout the book. The geometries come equipped with well-behaved distance functions, which in turn give rise to curvature of the space. The curved spaces in the title of this book will nearly always be two-dimensional, but this still enables us to study such basic geometrical ideas as geodesics, curvature and topology, and to understand how these ideas are interlinked. The classical examples will act both as an introduction to, and examples of, the more general theory of curved spaces studied later in the book, as represented by embedded surfaces in Euclidean 3-space, and more generally by abstract surfaces with Riemannian metrics.

The author has tried to make this text as self-contained as possible, although the reader will find it very helpful to have been exposed to first courses in Analysis, Algebra, and Complex Variables beforehand. The course is intended to act as a link between these basic undergraduate courses, and more theoretical geometrical theories, as represented say by courses on Riemann Surfaces, Differential Manifolds, Algebraic Topology or Riemannian Geometry. As such, the book is not intended to be another text on Differential Geometry, of which there are many good ones in the literature, but has rather different aims. For books on differential geometry, the author can recommend three in particular, which he has consulted when writing this volume, namely [5], [8] and [9]. The author has also not attempted to put the geometry he describes into a historical perspective, as for instance is done in [8].

As well as making the text as self-contained as possible, the author has tried to make it as elementary and as explicit as possible, where the use of the word elementary

here implies that we wish to rely as little as possible on theory developed elsewhere. This explicit approach does result in one proof where the general argument is both intuitive and clear, but where the specific details need care to get correct, the resulting formal proof therefore being a little long. This proof has been placed in an appendix to Chapter 3, and the reader wishing to maintain his or her momentum should skip over this on first reading. It may however be of interest to work through this proof at some stage, as it is by understanding where the problems lie that the more theoretical approach will subsequently be better appreciated. The format of the book has however allowed the author to be more expansive than was possible in the lectured course on certain other topics, including the important concepts of differentials and abstract surfaces. It is hoped that the latter parts of the book will also serve as a useful resource for more advanced courses in differential geometry, where our concrete approach will complement the usual rather more abstract treatments.

The author wishes to thank Nigel Hitchin for showing him the lecture notes of a course on Geometry of Surfaces he gave in Oxford (and previously given by Graeme Segal), which will doubtless have influenced the presentation that has been given here. He is grateful to Gabriel Paternain, Imre Leader and Dan Jane for their detailed and helpful comments concerning the exposition of the material, and to Sebastian Pancratz for his help with the diagrams and typesetting. Most importantly, he wishes to thank warmly his colleague Gabriel Paternain for the benefit of many conversations around the subject, which have had a significant impact on the final shape of the book.