## Contents

**Preface**

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 What it is all about?</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Motivation</td>
<td>2</td>
</tr>
<tr>
<td>1.3 Topologies and numerical methods</td>
<td>3</td>
</tr>
<tr>
<td>1.4 Choice of the control</td>
<td>4</td>
</tr>
<tr>
<td>1.5 Relaxation of the controllability notion</td>
<td>4</td>
</tr>
<tr>
<td>1.6 Various remarks</td>
<td>5</td>
</tr>
</tbody>
</table>

**Part I  Diffusion Models**

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Distributed and pointwise control for linear diffusion equations</td>
<td>9</td>
</tr>
<tr>
<td>1.1 First example</td>
<td>9</td>
</tr>
<tr>
<td>1.2 Approximate controllability</td>
<td>12</td>
</tr>
<tr>
<td>1.3 Formulation of the approximate controllability problem</td>
<td>14</td>
</tr>
<tr>
<td>1.4 Dual problem</td>
<td>15</td>
</tr>
<tr>
<td>1.5 Direct solution to the dual problem</td>
<td>17</td>
</tr>
<tr>
<td>1.6 Penalty arguments</td>
<td>19</td>
</tr>
<tr>
<td>1.7 $L^\infty$ cost functions and bang-bang controls</td>
<td>22</td>
</tr>
<tr>
<td>1.8 Numerical methods</td>
<td>28</td>
</tr>
<tr>
<td>1.9 Relaxation of controllability</td>
<td>57</td>
</tr>
<tr>
<td>1.10 Pointwise control</td>
<td>62</td>
</tr>
<tr>
<td>1.11 Further remarks (I): Additional constraints on the state function</td>
<td>96</td>
</tr>
<tr>
<td>1.12 Further remarks (II): A bisection based memory saving method for the solution of time dependent control problems by adjoint equation based methodologies</td>
<td>112</td>
</tr>
<tr>
<td>1.13 Further remarks (III): A brief introduction to Riccati equations based control methods</td>
<td>117</td>
</tr>
</tbody>
</table>
### Contents

**2 Boundary control**

2.1 Dirichlet control (I): Formulation of the control problem 124
2.2 Dirichlet control (II): Optimality conditions and dual formulations 126
2.3 Dirichlet control (III): Iterative solution of the control problems 128
2.4 Dirichlet control (IV): Approximation of the control problems 133
2.5 Dirichlet control (V): Iterative solution of the fully discrete dual problem (2.124) 143
2.6 Dirichlet control (VI): Numerical experiments 146
2.7 Neumann control (I): Formulation of the control problems and synopsis 155
2.8 Neumann control (II): Optimality conditions and dual formulations 163
2.9 Neumann control (III): Conjugate gradient solution of the dual problem (2.192) 176
2.10 Neumann control (IV): Iterative solution of the dual problem (2.208), (2.209) 178
2.11 Neumann control of unstable parabolic systems: a numerical approach 178
2.12 Closed-loop Neumann control of unstable parabolic systems via the Riccati equation approach 223

**3 Control of the Stokes system**

3.1 Generalities. Synopsis 231
3.2 Formulation of the Stokes system. A fundamental controllability result 231
3.3 Two approximate controllability problems 234
3.4 Optimality conditions and dual problems 234
3.5 Iterative solution of the control problem (3.19) 236
3.6 Time discretization of the control problem (3.19) 238
3.7 Numerical experiments 239

**4 Control of nonlinear diffusion systems**

4.1 Generalities. Synopsis 243
4.2 Example of a noncontrollable nonlinear system 243
4.3 Pointwise control of the viscous Burgers equation 245
4.4 On the controllability and the stabilization of the Kuramoto-Sivashinsky equation in one space dimension 259

**5 Dynamic programming for linear diffusion equations**

5.1 Introduction. Synopsis 277
5.2 Derivation of the Hamilton–Jacobi–Bellman equation 278
5.3 Some remarks 279
## Part II Wave Models

### 6 Wave equations

- 6.1 Wave equations: Dirichlet boundary control 283
- 6.2 Approximate controllability 285
- 6.3 Formulation of the approximate controllability problem 286
- 6.4 Dual problems 287
- 6.5 Direct solution of the dual problem 288
- 6.6 Exact controllability and new functional spaces 289
- 6.7 On the structure of space $E$ 291
- 6.8 Numerical methods for the Dirichlet boundary controllability of the wave equation 291
- 6.9 Experimental validation of the filtering procedure of Section 6.8.7 via the solution of the test problem of Section 6.8.5 315
- 6.10 Some references on alternative approximation methods 319
- 6.11 Other boundary controls 320
- 6.12 Distributed controls for wave equations 328
- 6.13 Dynamic programming 329

### 7 On the application of controllability methods to the solution of the Helmholtz equation at large wave numbers

- 7.1 Introduction 332
- 7.2 The Helmholtz equation and its equivalent wave problem 332
- 7.3 Exact controllability methods for the calculation of time-periodic solutions to the wave equation 334
- 7.4 Least-squares formulation of the problem (7.8)–(7.11) 334
- 7.5 Calculation of $J'$ 336
- 7.6 Conjugate gradient solution of the least-squares problem (7.14) 337
- 7.7 A finite element–finite difference implementation 340
- 7.8 Numerical experiments 341
- 7.9 Further comments. Description of a mixed formulation based variant of the controllability method 349
- 7.10 A final comment 355

### 8 Other wave and vibration problems. Coupled systems

- 8.1 Generalities and further references 356
- 8.2 Coupled Systems (I): a problem from thermo-elasticity 359
- 8.3 Coupled systems (II): Other systems 367
Part III  Flow Control

9  Optimal control of systems modelled by the Navier–Stokes equations: Application to drag reduction  371
  9.1  Introduction. Synopsis  371
  9.2  Formulation of the control problem  373
  9.3  Time discretization of the control problem  377
  9.4  Full discretization of the control problem  379
  9.5  Gradient calculation  384
  9.6  A BFGS algorithm for solving the discrete control problem  388
  9.7  Validation of the flow simulator  389
  9.8  Active control by rotation  394
  9.9  Active control by blowing and suction  408
  9.10  Further comments on flow control and conclusion  419

Epilogue  426

Further Acknowledgements  429

References  430

Index of names  450

Index of subjects  454