Part I

General relativity:
Classical studies of the Kerr geometry

1

The Kerr spacetime – a brief introduction

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1.1 Background

The Kerr spacetime has now been with us for some 45 years (Kerr 1963, 1965). It was discovered in 1963 through an intellectual *tour de force*, and continues to provide highly non-trivial and challenging mathematical and physical problems to this day. This chapter provides a brief introduction to the Kerr spacetime and rotating black holes, touching on the most common coordinate representations of the spacetime metric and the key features of the geometry – the presence of horizons and ergospheres. The coverage is by no means complete, and serves chiefly to orient oneself when reading subsequent chapters.

The final form of Albert Einstein's general theory of relativity was developed in November 1915 (Einstein 1915, Hilbert 1915), and within two months Karl Schwarzschild (working with one of the slightly earlier versions of the theory) had already solved the field equations that determine the exact spacetime geometry of a non-rotating "point particle" (Schwarzschild 1916a). It was relatively quickly realized, via Birkhoff's uniqueness theorem (Birkhoff 1923, Jebsen 1921, Deser and Franklin 2005, Johansen and Ravndal 2006), that the spacetime geometry in the vacuum region outside any localized spherically symmetric source is equivalent, up to a possible coordinate transformation, to a portion of the Schwarzschild geometry – and so of direct physical interest to modelling the spacetime geometry surrounding and exterior to idealized non-rotating spherical stars and planets. (In counterpoint, for modelling the *interior* of a finite-size spherically symmetric source, Schwarzschild's "constant density star" is a useful first approximation (Schwarzschild 1916b). This is often referred to as Schwarzschild's "interior" solution, which is potentially confusing as it is an utterly distinct physical

The Kerr Spacetime: Rotating Black Holes in General Relativity. Ed. David L. Wiltshire, Matt Visser and Susan M. Scott. Published by Cambridge University Press.

spacetime solving the Einstein equations in the presence of a specified distribution of matter.)

Considerably more slowly, only after intense debate was it realized that the "inward" analytic extension of Schwarzschild's "exterior" solution represents a non-rotating black hole, the endpoint of stellar collapse (Oppenheimer and Snyder 1939). In the most common form (Schwarzschild coordinates, also known as curvature coordinates), which is not always the most useful form for understanding the physics, the Schwarzschild geometry is described by the line element

$$\mathrm{d}s^2 = -\left[1 - \frac{2m}{r}\right] \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - 2m/r} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \; \mathrm{d}\phi^2), \tag{1.1}$$

where the parameter m is the physical mass of the central object.

But astrophysically, we know that stars (and for that matter planets) rotate, and from the weak-field approximation to the Einstein equations we even know the approximate form of the metric at large distances from a stationary isolated body of mass m and angular momentum J (Thirring and Lense 1918; Misner, Thorne, and Wheeler 1973; Adler, Bazin, and Schiffer 1975; D'Inverno 1992; Hartle 2003; Carroll 2004; Pfister 2005). In suitable coordinates:

$$\begin{aligned}\mathrm{d}s^2 = &-\left[1 - \frac{2m}{r} + O\left(\frac{1}{r^2}\right)\right] \mathrm{d}t^2 - \left[\frac{4J\sin^2\theta}{r} + O\left(\frac{1}{r^2}\right)\right] \mathrm{d}\phi \; \mathrm{d}t \\ &+ \left[1 + \frac{2m}{r} + O\left(\frac{1}{r^2}\right)\right] [\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \; \mathrm{d}\phi^2)].\end{aligned} \tag{1.2}$$

This approximate metric is perfectly adequate for almost all solar system tests of general relativity, but there certainly are well-known astrophysical situations (e.g. neutron stars) for which this approximation is inadequate – and so a "strong field" solution is physically called for. Furthermore, if a rotating star were to undergo gravitational collapse, then the resulting black hole would be expected to retain at least some fraction of its initial angular momentum – thus suggesting on physical grounds that somehow there should be an extension of the Schwarzschild geometry to the situation where the central body carries angular momentum.

Physicists and mathematicians looked for such a solution for many years, and had almost given up hope, until the Kerr solution was discovered in 1963 (Kerr 1963) – some 48 years after the Einstein field equations were first developed. From the weak-field asymptotic result we can already see that angular momentum destroys spherical symmetry, and this lack of spherical symmetry makes the calculations *much* more difficult. It is not that the basic principles are all that different, but simply that the algebraic complexity of the computations is so high that relatively few physicists or mathematicians have the fortitude to carry them through to completion.

Indeed it is easy to both derive and check the Schwarzschild solution by hand, but for the Kerr spacetime the situation is rather different. For instance in Chandrasekhar's magnum opus on black holes (Chandrasekhar 1998), only part of which is devoted to the Kerr spacetime, he is moved to comment:

> The treatment of the perturbations of the Kerr space-time in this chapter has been prolixius in its complexity. Perhaps, at a later time, the complexity will be unravelled by deeper insights. But meantime, the analysis has led us into a realm of the rococo: splendorous, joyful, and immensely ornate.

More generally, Chandrasekhar also comments:

> The nature of developments simply does not allow a presentation that can be followed in detail with modest effort: the reductions that are required to go from one step to another are often very elaborate and, on occasion, may require as many as ten, twenty, or even fifty pages.

Of course the Kerr spacetime was not constructed *ex nihilo*. Some of Roy Kerr's early thoughts on this and related matters can be found in (Kerr 1959; Kerr and Goldberg 1961), and over the years he has periodically revisited this theme (Goldberg and Kerr 1964; Kerr and Schild 1965; Debney, Kerr, and Schild 1969; Kerr and Debney 1970; Kerr and Wilson 1977; Weir and Kerr 1977; Fackerell and Kerr 1991; Burinskii and Kerr 1995).

For practical and efficient computation in the Kerr spacetime many researchers will prefer to use general symbolic manipulation packages such as Maple, Mathematica, or more specialized packages such as GR-tensor. When used with *care* and *discretion*, symbolic manipulation tools can greatly aid physical understanding and insight.[1]

Because of the complexity of calculations involving the Kerr spacetime there is relatively little textbook coverage dedicated to this topic. An early discussion can be found in the textbook by Adler, Bazin, and Schiffer (1975 second edition). The only dedicated single-topic textbook I know of is that by O'Neill (1995). There are also comparatively brief discussions in the research monograph by Hawking and Ellis (1975), and the standard textbooks by Misner, Thorne, and Wheeler (1973), D'Inverno (1992), Hartle (2003), and Carroll (2004). One should particularly note the 60-page chapter appearing in the very recent textbook by Plebański and Krasiński (2006). An extensive and highly technical discussion of Kerr black holes is given in Chandrasekhar (1998), while an exhaustive discussion of the class of spacetimes described by Kerr–Schild metrics is presented in the book "Exact Solutions to Einstein's Field Equations" (Stephani *et al*., 2002).

[1] For instance, the standard distribution of Maple makes some unusual choices for its sign conventions. The signs of the Einstein tensor, Ricci tensor, and Ricci scalar (though *not* the Riemann tensor and Weyl tensor) are opposite to what most physicists and mathematicians would expect.

To orient the reader I will now provide some general discussion, and explicitly present the line element for the Kerr spacetime in its most commonly used coordinate systems. (Of course the physics cannot depend on the coordinate system, but specific computations can sometimes be simplified by choosing an appropriate coordinate chart.)

1.2 No Birkhoff theorem

Physically, it must be emphasized that there is no Birkhoff theorem for rotating spacetimes – it is *not* true that the spacetime geometry in the vacuum region outside a generic rotating star (or planet) is a part of the Kerr geometry. The best result one can obtain is the much milder statement that outside a rotating star (or planet) the geometry asymptotically approaches Kerr geometry.

The basic problem is that in the Kerr geometry all the multipole moments are very closely related to each other – whereas in real physical stars (or planets) the mass quadrupole, octopole, and higher moments of the mass distribution can in principle be independently specified. Of course from electromagnetism you will remember that higher n-pole fields fall off as $1/r^{2+n}$, so that far away from the object the lowest multipoles dominate – it is in this *asymptotic* sense that the Kerr geometry is relevant for rotating stars or planets.

On the other hand, if the star (or planet) gravitationally collapses – then classically a black hole can be formed. In this case there *are* a number of powerful uniqueness theorems which guarantee the direct physical relevance of the Kerr spacetime, but as the unique exact solution corresponding to stationary rotating black holes, (as opposed to merely being an asymptotic solution to the far field of rotating stars or planets).

1.3 Kerr's original coordinates

The very first version of the Kerr spacetime geometry to be explicitly written down in the literature was the line element (Kerr 1963)

$$\begin{aligned} \mathrm{d}s^2 = &- \left[1 - \frac{2mr}{r^2 + a^2 \cos^2\theta} \right] (\mathrm{d}u + a \sin^2\theta \; \mathrm{d}\phi)^2 \\ &+ 2(\mathrm{d}u + a \sin^2\theta \; \mathrm{d}\phi)\,(\mathrm{d}r + a \sin^2\theta \; \mathrm{d}\phi) \\ &+ (r^2 + a^2\cos^2\theta)\,(\mathrm{d}\theta^2 + \sin^2\theta \; \mathrm{d}\phi^2). \end{aligned} \tag{1.3}$$

The key features of this spacetime geometry are:

- Using symbolic manipulation software it is easy to verify that this manifold is Ricci flat, $R_{ab} = 0$, and so satisfies the vacuum Einstein field equations. Verifying this by hand is at best tedious.
- There are three off-diagonal terms in the metric – which is one of the features that makes computations tedious.
- By considering (for instance) the g_{uu} component of the metric, it is clear that for $m \neq 0$ there is (at the very least) a coordinate singularity located at $r^2 + a^2 \cos^2\theta = 0$, that is:

$$r = 0; \qquad \theta = \pi/2. \tag{1.4}$$

  We shall soon see that this is actually a curvature singularity. In these particular coordinates there are no other obvious coordinate singularities.
- Since the line element is independent of both u and ϕ we immediately deduce the existence of two Killing vectors. Ordering the coordinates as (u, r, θ, ϕ) the two Killing vectors are

$$U^a = (1, 0, 0, 0); \qquad R^a = (0, 0, 0, 1). \tag{1.5}$$

  Any constant-coefficient linear combination of these Killing vectors will again be a Killing vector.
- Setting $a \to 0$ the line element reduces to

$$\mathrm{d}s^2 \to -\left[1 - \frac{2m}{r}\right] \mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r + r^2\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{1.6}$$

  which is the Schwarzschild geometry in the so-called "advanced Eddington–Finkelstein coordinates". Based on this, by analogy the line element (1.3) is often called the advanced Eddington–Finkelstein form of the Kerr spacetime. Furthermore since we know that $r = 0$ is a curvature singularity in the Schwarzschild geometry, this strongly suggests (but does not yet prove) that the singularity in the Kerr spacetime at $(r = 0, \theta = \pi/2)$ is a curvature singularity.
- Setting $m \to 0$ the line element reduces to

$$\begin{aligned} \mathrm{d}s^2 \to \mathrm{d}s_0^2 = &-(\mathrm{d}u + a\sin^2\theta\,\mathrm{d}\phi)^2 \\ &+ 2(\mathrm{d}u + a\sin^2\theta\,\mathrm{d}\phi)\,(\mathrm{d}r + a\sin^2\theta\,\mathrm{d}\phi) \\ &+ (r^2 + a^2\cos^2\theta)\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \end{aligned} \tag{1.7}$$

  which is actually (*but certainly not obviously*) flat Minkowski spacetime in disguise. This is most easily seen by using symbolic manipulation software to verify that for this simplified line element the Riemann tensor is identically zero: $R_{abcd} \to 0$.
- For the general situation, $m \neq 0 \neq a$, all the non-zero components of the Riemann tensor contain at least one factor of m.

- Indeed, in a suitably chosen orthonormal basis the result can be shown to be even stronger: All the non-zero components of the Riemann tensor are then proportional to m:

$$R_{\hat{a}\hat{b}\hat{c}\hat{d}} \propto m. \tag{1.8}$$

(This point will be discussed more fully below, in the section on the rational polynomial form of the Kerr metric. See also the discussion in Plebański and Krasiński (2006).)

- Furthermore, the only non-trivial quadratic curvature invariant is

$$\begin{aligned} R_{abcd}\, R^{abcd} &= C_{abcd}\, C^{abcd} \\ &= \frac{48m^2(r^2 - a^2\cos^2\theta)\,[(r^2 + a^2\cos^2\theta)^2 - 16r^2a^2\cos^2\theta]}{(r^2 + a^2\cos^2\theta)^6}, \end{aligned} \tag{1.9}$$

guaranteeing that the singularity located at

$$r = 0; \qquad \theta = \pi/2, \tag{1.10}$$

is actually a curvature singularity. (We would have strongly suspected this by considering the $a \to 0$ case above.)

- In terms of the $m = 0$ line element we can put the line element into manifestly Kerr–Schild form by writing

$$\mathrm{d}s^2 = \mathrm{d}s_0^2 + \frac{2mr}{r^2 + a^2\cos^2\theta}\,(\mathrm{d}u + a\sin^2\theta\,\mathrm{d}\phi)^2, \tag{1.11}$$

or the equivalent

$$g_{ab} = (g_0)_{ab} + \frac{2mr}{r^2 + a^2\cos^2\theta}\,\ell_a\,\ell_b, \tag{1.12}$$

where we define

$$(g_0)_{ab} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & a\sin^2\theta \\ 0 & 0 & r^2 + a^2\cos^2\theta & 0 \\ 0 & a\sin^2\theta & 0 & (r^2 + a^2)\sin^2\theta \end{bmatrix}, \tag{1.13}$$

and

$$\ell_a = (1, 0, 0, a\sin^2\theta). \tag{1.14}$$

- It is then easy to check that ℓ_a is a null vector, with respect to both g_{ab} and $(g_0)_{ab}$, and that

$$g^{ab} = (g_0)^{ab} - \frac{2mr}{r^2 + a^2\cos^2\theta}\,\ell^a\,\ell^b, \tag{1.15}$$

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where

$$(g_0)^{ab} = \frac{1}{r^2 + a^2 \cos^2\theta} \begin{bmatrix} a^2 \sin^2\theta & r^2 + a^2 & 0 & -a \\ r^2 + a^2 & r^2 + a^2 & 0 & -a \\ 0 & 0 & 1 & 0 \\ -a & -a & 0 & (\sin^2\theta)^{-1} \end{bmatrix}, \tag{1.16}$$

and

$$\ell^a = (0, 1, 0, 0). \tag{1.17}$$

- The determinant of the metric takes on a remarkably simple form

$$\det(g_{ab}) = -(r^2 + a^2\ \cos^2\theta)^2 \sin^2\theta = \det([g_0]_{ab}), \tag{1.18}$$

where the m dependence has cancelled. (This is a side effect of the fact that the metric is of the Kerr–Schild form.)

- At the curvature singularity we have

$$\mathrm{d}s_0^2\big|_{\text{singularity}} = -\mathrm{d}u^2 + a^2\ \mathrm{d}\phi^2, \tag{1.19}$$

showing that, in terms of the "background" geometry specified by the disguised Minkowski spacetime with metric $(g_0)_{ab}$, the curvature singularity is a "ring". Of course in terms of the "full" geometry, specified by the physical metric g_{ab}, the intrinsic geometry of the curvature singularity is, unavoidably and by definition, singular.

- The null vector field ℓ^a is an affinely parameterized null geodesic:

$$\ell^a\ \nabla_a \ell^b = 0. \tag{1.20}$$

- More generally

$$\ell_{(a;b)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin^2\theta \end{bmatrix}_{ab} - \frac{m(r^2 - a^2\cos^2\theta)}{(r^2 + a^2\cos^2\theta)^2}\ \ell_a\ \ell_b. \tag{1.21}$$

(And it is easy to see that this automatically implies the null vector field ℓ^a is an affinely parameterized null geodesic.)

- The divergence of the null vector field ℓ^a is also particularly simple

$$\nabla_a \ell^a = \frac{2r}{r^2 + a^2\ \cos^2\theta}. \tag{1.22}$$

- Furthermore, with the results we already have it is easy to calculate

$$\ell_{(a;b)}\ \ell^{(a;b)} = \ell_{(a;b)} g^{bc} \ell_{(c;d)} g^{da} = \ell_{(a;b)} [g_0]^{bc} \ell_{(c;d)} [g_0]^{da} = \frac{2r^2}{(r^2 + a^2\cos^2\theta)^2}, \tag{1.23}$$

whence

$$\frac{\ell_{(a;b)}\,\ell^{(a;b)}}{2} - \frac{(\nabla_a \ell^a)^2}{4} = 0. \tag{1.24}$$

This invariant condition implies that the null vector field ℓ^a is “shear free”.

- Define a one-form ℓ by

$$\ell = \ell_a\, \mathrm{d}x^a = \mathrm{d}u + a\sin^2\theta\, \mathrm{d}\phi, \tag{1.25}$$

then

$$\mathrm{d}\ell = a\sin 2\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi \neq 0, \tag{1.26}$$

but implying

$$\mathrm{d}\ell \wedge \mathrm{d}\ell = 0. \tag{1.27}$$

- Similarly

$$\ell \wedge \mathrm{d}\ell = a\sin 2\theta\, \mathrm{d}u \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi \neq 0, \tag{1.28}$$

but in terms of the Hodge-star we have the invariant relation

$$*(\ell \wedge \mathrm{d}\ell) = -\frac{2a\cos\theta}{r^2 + a^2\cos^2\theta}\,\ell, \tag{1.29}$$

or in component notation

$$\epsilon^{abcd}(\ell_b\,\ell_{c,d}) = -\frac{2a\cos\theta}{r^2 + a^2\cos^2\theta}\,\ell^a. \tag{1.30}$$

This allows one to pick off the so-called “twist” as

$$\omega = -\frac{a\cos\theta}{r^2 + a^2\cos^2\theta}. \tag{1.31}$$

This list of properties is a quick, but certainly not exhaustive, survey of the key features of the spacetime that can be established by direct computation in this particular coordinate system.

1.4 Kerr–Schild “Cartesian” coordinates

The second version of the Kerr line element presented in the original article (Kerr 1963), also discussed in the early follow-up conference contribution (Kerr 1965),

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was defined in terms of "Cartesian" coordinates (t, x, y, z):

$$\begin{aligned} ds^2 = &-dt^2 + dx^2 + dy^2 + dz^2 \\ &+ \frac{2mr^3}{r^4 + a^2z^2}\left[dt + \frac{r(x\ dx + y\ dy)}{a^2 + r^2} + \frac{a(y\ dx - x\ dy)}{a^2 + r^2} + \frac{z}{r}\,dz\right]^2, \end{aligned} \tag{1.32}$$

subject to $r(x, y, z)$, which is now a dependent function, not a coordinate, being implicitly determined by:

$$x^2 + y^2 + z^2 = r^2 + a^2\left[1 - \frac{z^2}{r^2}\right]. \tag{1.33}$$

- The coordinate transformation used in going from (1.3) to (1.32) is

$$t = u - r; \qquad x + iy = (r - ia)\, e^{i\phi} \sin\theta; \qquad z = r\cos\theta. \tag{1.34}$$

  Sometimes it is more convenient to explicitly write

$$x = (r\cos\phi + a\sin\phi)\sin\theta = \sqrt{r^2 + a^2}\ \sin\theta\cos[\phi - \arctan(a/r)]; \tag{1.35}$$
$$y = (r\sin\phi - a\cos\phi)\sin\theta = \sqrt{r^2 + a^2}\ \sin\theta\sin[\phi - \arctan(a/r)]; \tag{1.36}$$

  and so deduce

$$\frac{x^2 + y^2}{\sin^2\theta} - \frac{z^2}{\cos^2\theta} = a^2, \tag{1.37}$$

  or the equivalent

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1. \tag{1.38}$$

- The $m \to 0$ limit is now manifestly Minkowski space

$$ds^2 \to ds_0^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{1.39}$$

  Of course the coordinate transformation (1.34) used in going from (1.3) to (1.32) is also responsible for taking (1.7) to (1.39).
- The $a \to 0$ limit is

$$ds^2 \to -dt^2 + dx^2 + dy^2 + dz^2 + \frac{2m}{r}\left[dt + \frac{(x\ dx + y\ dy + z\ dz)}{r}\right]^2, \tag{1.40}$$

  now with

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{1.41}$$