Introduction

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This book has some of its genesis in the, possibly apocryphal, story that at an acoustics conference in the late 1980s a certain distinguished professor, tiring of the proceedings, turned to the assembled researchers and announced

> Listen! If what you're doing isn't nonlinear or transonic, then don't bother! It's all been done!

Certainly it has become easy to think of linear acoustics as essentially completed. After all, classic texts such as Morse and Feshbach (1953) give admirably thorough expositions of very general techniques, particularly those based on Green's functions. Cases described by coordinate systems in which the governing equations are separable are extensively tabulated and admit analytic solutions. The alternative is to employ numerical methods, many of them also based on Green's functions, which work in arbitrarily complex geometries. There is perhaps a perception that notwithstanding a host of important applied problems, there are no fundamental issues remaining in linear acoustics. Increased understanding of the richness and complexity of nonlinear problems with the explosion of interest in chaos only serves to make linear systems seem "done and dusted" in comparison.

And yet this picture is overly dismissive. A solution of a linear differential equation depends nonlinearly on its coefficients and the shape of the boundary. The dependence is all the richer if those coefficients are random or if boundary reflections are defocusing. Developments in physics throughout the last four decades, often equally applicable to both quantum and linear acoustic problems, but overwhelmingly more often expressed in the language of the former, have explored this. More than that they have provided a new way of thinking about such things. We have been impressed at the significant new body of theory that can be used to address problems in linear acoustics and vibration, although also disappointed at the small amount of reported work that does so. This book is an attempt to bridge the gap between theoreticians and practitioners, as well as the gap between quantum and acoustic, a gap that is mostly terminological but should nevertheless not be underestimated. Our hope is that acousticians and vibration engineers who wish to see what can be done with these new tools will find in this book a comprehensible introduction and that physicists may also learn what problems might usefully be addressed.

So what is on offer? We would like to take the reader on a short guided tour of the terrain. We begin with what is known as the *semiclassical trace formula* (Chapter 1), which expresses the modal density of a closed, lossless enclosure (membrane or cavity) in terms of its *periodic orbits*, closed internal ray paths that repeat indefinitely. As a way to determine eigenvalues (let alone response to arbitrary excitations) it cannot compete with the numerical techniques that have been refined for use in engineering (such as finite elements) or physics (such as plane-wave decomposition); its significance lies in the fact that it provides an explicit link between the shape of an enclosure and its acoustic characteristics, both in an average sense (via the Weyl series) and at the level of individual eigenvalues, and in a way that doesn't depend on separability.

This connection is important because for many shapes the periodic orbits are unstable and the ray paths are *chaotic*, the implications of which are explored in Chapter 2. It can be disconcerting to find chaos having such a profound influence on linear systems. This is due to the nonlinearity of ray motion in the high-frequency limit, and the study of the effects of this on the finite-frequency wave motion has come to be known as quantum chaology or (despite linguistic objections) *quantum chaos*. It used to be easy to imagine that almost all ordinary differential equations had well-behaved, predictable solutions because almost all the ones in books did. That misapprehension was shattered by the explosion of awareness about chaos. In the same way it is easy to fall into the trap of thinking that modeshapes and natural frequencies are as simple and regular in arbitrary shapes as those of the simple textbook examples used to teach the subject. They are not, and for very similar reasons.

One of the consequences of chaotic ray motion is that eigenfunctions often resemble superpositions of Gaussian random waves, the properties of which are explored in more detail in Chapter 4. Those that do not are referred to as "scarred modes"; Chapter 5 presents an ingenious formulation that allows the eigenfunctions to be represented with impressive efficiency in a basis built out of deliberately constructed scar functions. Of course acousticians rarely encounter truly lossless systems in practice; so some of the implications of opening the enclosure are explored in Chapter 6. And in Chapter 7 the central result of the periodic orbit theory is re-derived in a form suitable for elasticity so as to expand the range of possible applications.

Before that, however, we introduce the second major theme of this book: *random matrix theory*. The study of the statistics of the eigenvalues of ensembles of matrices whose elements are random variables and exhibit a particular symmetry began in nuclear physics as an exploration of the conjecture that a sufficiently complex system might have properties statistically similar to those of a random Hamiltonian. Modern computational capabilities have made it easier to test conjectures and confirm analytic results. For example, the fact that the normalized spacings of the eigenvalues of a large Gaussian Orthogonal matrix are close to the Rayleigh distribution (obeyed exactly by an ensemble of pairs of eigenvalues of 2×2 Gaussian orthogonal matrices) can be shown using less than 10 lines of

MATLAB† and can be computed in a few seconds. Chapter 3 introduces the theory that allows such predictions and, as its name implies, explores why such an approach should be so effective in describing the behavior of the wave-bearing and vibrating systems we are considering here.

Our third theme, *complexity* does not get a chapter to itself or even an index entry. Instead it is embedded throughout the book in the richness of the behavior of simple systems and the diversity of applications in the later chapters. Each reader will make their own connections between the various topics here, but one striking example is worth noting here: how in a multitude of contexts "the part contains the whole." Just as each cell of an organism contains the DNA of the whole being, a few short periodic orbits contain information about a large part of the eigenstructure; in seismology and underwater acoustics a short part of a time history reveals information about the whole system.

Subsequent chapters survey several applied topics related in varying degrees to the earlier chapters. Inasmuch as multiple scattering plays such a recurrent and important role in mesoscopics (the subject of Chapter 8), we also include a review of the, often too obscure to the non-initiate, diagrammatic methods for the theory of randomly scattered acoustics in Chapter 9. The surprising and highly applicable results of the theory of time-reversed waves are explored in Chapter 10 with particular reference to the themes of this book, which have led to important applications in ultrasonics.

Chapter 11 shows the relevance of ray chaos for long-range propagation in the ocean, whereas Chapter 12 demonstrates applications in seismology. Chapter 13 shows how random matrix theory can be applied to structural acoustics and vibrations, whereas Chapter 14 explains an alternative random matrix theory approach to the problem of estimating the likely variation in response that results from the inevitable small variations that arise in manufacturing.

It is impossible in a book of practical length to cover all the modern applications of these ideas that we might have, and we apologize to those who have noted holes in our coverage. Perhaps there will be a need for another book.

As editors we wish to thank the authors and the publishers for their patience during the unfortunately long time it has taken to turn their contributions into this book. We express our gratitude to all the publishers who granted permission for the chapter authors to reuse figures from their published articles without payment, and our greater gratitude to those who provided it as a matter of policy without being asked. We have tried to attribute all reused figures; if we have inadvertently failed to do so we would be grateful to be informed and will endeavor to correct the

† For the avoidance of doubt they are as follows:

```
n = 2000;
A = randn(n);
E = eig((A + A')/2);
s = diff(E).*real(sqrt(2*n - E(1:n-1).^2)/pi);
[N,x] = hist(s,40);
bar(x,N/n/(x(2) - x(1)))
hold on
plot(x,(pi/2)*x.*exp((-pi/4)*x.^2),'r','LineWidth',2)
hold off
```

oversight in future editions if there are any. We also wish to thank the organizers of the 2005 Summer School on Chaotic and Random Wave Scattering at the Centro International de Ciencias A. C. in Cuernavaca, at which the idea for this book was born when we accidentally got separated from the rest of our party while exploring the pyramids of Xochicalco and their notable acoustics.

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1 The Semiclassical Trace Formula

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1.1 Introduction

For a two-dimensional enclosure, such as a membrane or the cross section of an infinitely long duct, those with the very simplest shapes (circles, rectangles, spheres, boxes, etc.) with simple uniform boundary conditions, the modes and natural frequencies can be determined analytically. For any other shape they may be determined numerically by a range of mature numerical techniques of which finite element and boundary element analyses are the best known and the most widely studied. Knowing how to calculate the modes and natural frequencies for any particular shape, however, is not the same as understanding how those modes and natural frequencies depend on the shape. Suppose, for example, that we wish to improve the design of a component by optimizing some quantity such as weight, while leaving its natural frequencies unchanged. In the course of such an optimization changes will be made to the shape, whereupon the process of calculating the modes and natural frequencies must begin all over again; at best, part of the mesh can be re-used. Such an analysis cannot tell us where effort can be most or least profitably concentrated.

It turns out that the shapes that can be analyzed are (for good reason) quite untypical compared with arbitrary shapes. The situation mirrors the one that used to prevail in the study of dynamical systems, where linear differential equations were most widely studied because of their solubility, and the fact that other systems showed radically different qualitative behavior was, for a time, ignored. In both cases the overlooked feature is chaos, but in the case of acoustic morphology the phenomenon is known as quantum chaos. Despite its name, this phenomenon can be exhibited by large-scale systems such as acoustical resonators, whose governing equations are entirely linear. It arises when a ray path is unstable to small perturbations and displays strong sensitivity to initial conditions.

Several surveys (Berry 1987, Guhr et al. 1998, Galdi et al. 2005, Kuhl et al. 2005) and books (Gutzwiller 1990, Ott 1993, Brack & Bhaduri 1997, Stöckmann 1999, Richter 2000, Haake 2001, Nakamura & Harayama 2004, Reichl 2004, Cvitanović et al. 2005) on aspects of this subject have become available in recent years, but these are variously intended for physicists, mathematicians, and electronic engineers. The theory of periodic orbits, and of quantum chaos, is applicable to a far greater range of areas than just acoustics, and naturally these texts span that range.

1.2 Introductory Examples

1.2.1 Modes in a Rectangular Enclosure

The rectangle is perhaps the simplest case to study because an explicit formula exists for its natural frequencies. From here on we shall work with wavenumber rather than frequency, and so we shall use the equation for the eigenwavenumbers of a rectangle with sides a_1, a_2:

$$k_{n,m} = \pi\left(\frac{n^2}{a_1^2} + \frac{m^2}{a_2^2}\right)^{1/2}, \tag{1.1}$$

where the indices n and m run $0, 1, 2, \ldots$ for Neumann boundary conditions and $1, 2, 3, \ldots$ for Dirichlet conditions. The spectral density of this system is defined as

$$\rho(k) = \sum_{n,m} \delta(k - k_{n,m}) \tag{1.2}$$

and the modecount as

$$N(k) = \int_0^k \rho(k')\,\mathrm{d}k' = \sum_{n,m} \mathrm{H}(k - k_{n,m}), \tag{1.3}$$

where H is the Heaviside function. We shall now show how alternative, series-form expressions for $\rho(k)$ and $N(k)$ can be obtained.

The delta functions in (1.2) can be written as the limit of a Gaussian function

$$\delta(k - k_{n,m}) = \lim_{t\to 0} \frac{1}{2\sqrt{\pi t}} \mathrm{e}^{-(k-k_{n,m})^2/4t}. \tag{1.4}$$

We can therefore write the spectral density function in the form

$$\rho(k) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lim_{t\to 0} \frac{1}{2\sqrt{\pi t}} \mathrm{e}^{-\left(k-\pi\sqrt{n^2/a_1^2+m^2/a_2^2}\right)^2/4t}. \tag{1.5}$$

The Poisson formula for a double sum,

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(n,m) = &\sum_{M_1=-\infty}^{\infty}\sum_{M_2=-\infty}^{\infty} \iint_0^{\infty} f(n_1,n_2)\mathrm{e}^{2\pi\mathrm{i}(M_1n_1+M_2n_2)}\,\mathrm{d}n_1\,\mathrm{d}n_2 \\ &+\frac{1}{2}\sum_{M_1=-\infty}^{\infty}\int_0^{\infty} f(n_1,0)\mathrm{e}^{2\pi\mathrm{i}M_1n_1}\mathrm{d}n_1 \\ &+\frac{1}{2}\sum_{M_2=-\infty}^{\infty}\int_0^{\infty} f(0,n_2)\mathrm{e}^{2\pi i M_2n_2}\mathrm{d}n_2 \\ &+\frac{1}{4}f(0,0),\end{aligned} \tag{1.6}$$

can be applied to (1.5). We shall take each term separately, denoting them F_1, F_2, F_3, F_4.

The expression for F_1 can be integrated by making the substitutions

$$n_1 = \frac{a_1 r}{\pi}\cos\theta, \qquad n_2 = \frac{a_2 r}{\pi}\sin\theta, \qquad \mathrm{d}n_1\,\mathrm{d}n_2 = \frac{a_1a_2}{\pi^2} r\,\mathrm{d}r\,\mathrm{d}\theta, \tag{1.7}$$

giving

$$F_1 = \sum_{M_1=-\infty}^{\infty} \sum_{M_2=-\infty}^{\infty} \int_0^{\infty} \int_0^{\pi/2} \lim_{t\to 0} \frac{a_1 b_1}{\pi^2} \frac{1}{2\sqrt{\pi t}} \mathrm{e}^{-(k-r)^2/4t+2\mathrm{i}(M_1 a_1 \cos\theta + M_2 a_2 \sin\theta) r} r \,\mathrm{d}r \,\mathrm{d}\theta. \tag{1.8}$$

After some manipulation this gives

$$F_1 = \frac{a_1 a_2 k}{2\pi} \sum_{M_1=-\infty}^{\infty} \sum_{M_2=-\infty}^{\infty} \mathrm{J}_0(k L_{M_1,M_2}), \tag{1.9}$$

where $L_{M_1,M_2} = 2\sqrt{M_1^2 a_2^2 + M_2^2 a_2^2}$ and J_0 is a Bessel function of zero order.

For F_2 we have

$$\begin{aligned} F_2 &= -\frac{1}{2} \sum_{M_1=-\infty}^{\infty} \lim_{t\to 0} \frac{1}{2\sqrt{\pi t}} \int_0^{\infty} \mathrm{e}^{-(k-\pi n_1/a_1)^2/4t+2\pi \mathrm{i} M_1 n_1} \mathrm{d}n_1 \\ &= -\frac{1}{2} \sum_{M_1=-\infty}^{\infty} \lim_{t\to 0} \frac{a_1}{2\pi} \mathrm{e}^{2M_1 a_1(\mathrm{i}k - 2M_1 a_1 t)} \left[1 + \mathrm{erf}\left(\frac{k + 4\mathrm{i} M_1 a_1 t}{2\sqrt{t}}\right)\right] \\ &= -\frac{a_1}{2\pi} \sum_{M_1=-\infty}^{\infty} \mathrm{e}^{2\mathrm{i}k M_1 a_1} \\ &= -\frac{a_1}{2\pi} \sum_{M_1=-\infty}^{\infty} \cos(2k M_1 a_1), \end{aligned} \tag{1.10}$$

and F_3 is the same with all subscripts 1 changed to 2 throughout. It can be shown that taking the sums on the left-hand side of (1.6) from 1 instead of 0, which would correspond to Neumann, rather than Dirichlet, boundary conditions, would reverse the sign of F_2 and F_3.

We therefore have

$$\rho(k) = \frac{a_1 a_2 k}{2\pi} \sum_{M_1,M_2=-\infty}^{\infty} \mathrm{J}_0(k L_{M_1,M_2}) \pm \sum_{i=1,2} \sum_{M=-\infty}^{\infty} \frac{a_i}{2\pi} \cos(2k M a_i) + \frac{\delta(k)}{4} \tag{1.11}$$

for Dirichlet (Neumann), conditions. Figure 1.1 shows a series of ray paths drawn in the rectangular domain, which reflect M_1 and M_2 times from the left and bottom walls, respectively, before returning to their origin with the initial heading so as to be able to repeat indefinitely. Such closed paths are called *periodic orbits*. Their length is given by L_{M_1,M_2}. This is no coincidence, as will be seen. The term $2kMa_i$ that forms the argument of the cosine in the second term can also be interpreted as the length of a ray path traveling between two parallel sides.

Because $\rho(k)$ is singular for all $k = k_n$, it must be smoothed before evaluation. In practice, we find it more convenient to work with $N(k)$, its integral with respect to k. Before evaluating this, however, we shall separate out the terms corresponding to zero-length orbits as

$$\overline{\rho}(k) = \frac{a_1 a_2}{2\pi} k \pm \frac{a_1 + a_2}{2\pi} + \frac{\delta(k)}{4}, \tag{1.12}$$

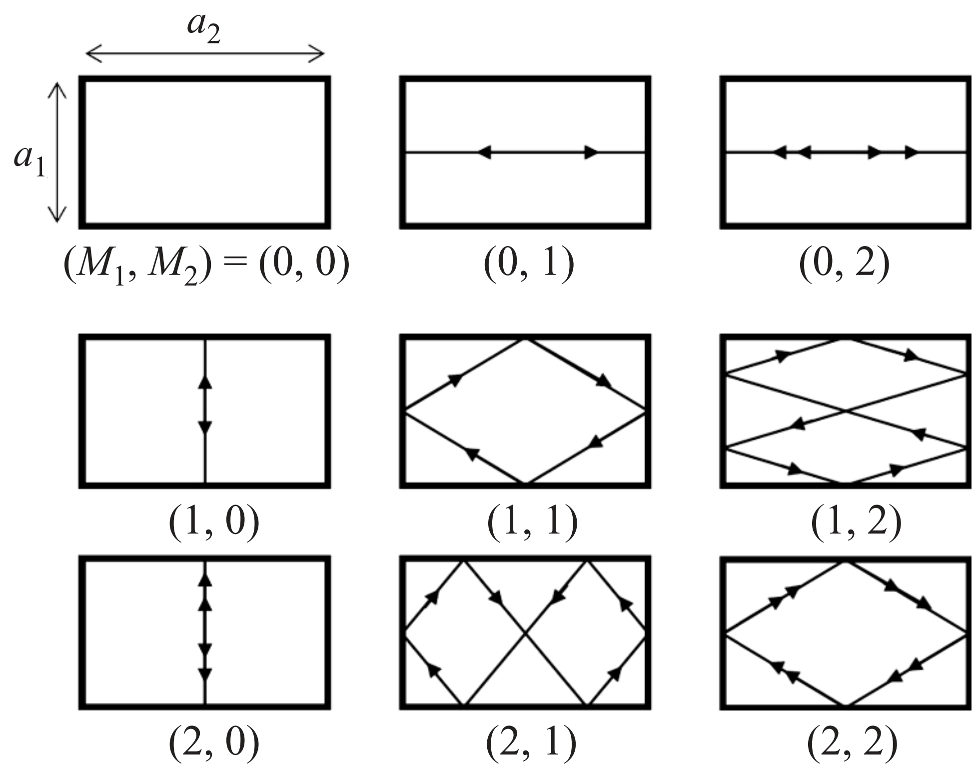

Figure 1.1. Periodic orbits for a rectangular enclosure.

leaving the remainder

$$\rho_{\text{osc}}(k) = \frac{a_1 a_2 k}{2\pi} \sum_{M_1, M_2=-\infty}^{\infty}{}' J_0(k L_{M_1,M_2}) \pm \sum_{i=1,2} \sum_{M=-\infty}^{\infty}{}' \frac{a_i}{2\pi} \cos(2k M a_i), \tag{1.13}$$

where the primes on the summations indicate that the terms in which all indices are zero are omitted. The smooth components can be integrated to give

$$\begin{aligned} \overline{N}(k) &= \frac{a_1 a_2}{4\pi} k^2 \mp \frac{a_1 + a_2}{2\pi} + \frac{1}{4} \\ &= \frac{A}{4\pi} k^2 \mp \frac{L}{4\pi} k + \frac{1}{4}, \end{aligned}$$

which is the well-known formula for the average number of modes in a rectangular enclosure with area A and perimeter L (see, e.g., Morse & Ingard 1968). The oscillating component can also be integrated to give

$$N_{\text{osc}}(k) = \frac{a_1 a_2 k}{2\pi} \sum_{M_1, M_2=-\infty}^{\infty}{}' \frac{J_1(k L_{M_1,M_2})}{L_{M_1,M_2}} \pm \sum_{i=1,2} \sum_{M_i=-\infty}^{\infty}{}' \frac{\sin(2k M_i a_i)}{4\pi M}, \tag{1.14}$$

where the second term can be recognized as the Fourier series representation of a sawtooth wave.

Partial sums of (1.14) plus $\overline{N}(k)$ are compared with the true modecount, calculated by evaluating (1.3) explicitly, in Figure 1.2.

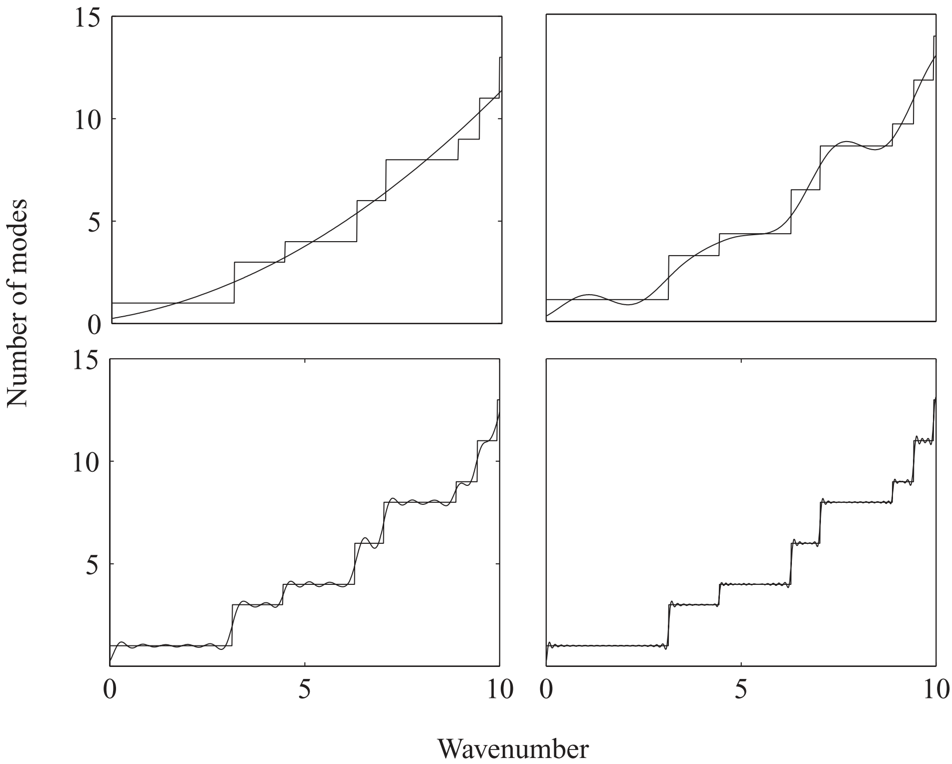

Figure 1.2. Partial sums of the semiclassical approximation to the modecount for a rectangular membrane with maximum values of M_i in all the summations of 0, 1, 4, and 20 respectively. After Wright (2001). Copyright 2001, the Acoustical Society of America.

1.2.2 The Length Spectrum of a Circle

Rather than try to derive a similar formula for the circle we will, for now, conjecture that such a formula exists and that it is of the form

$$\rho(k) \approx \sum_{\mathrm{PO}} A_{\mathrm{PO}}(k) \cos(k L_{\mathrm{PO}} + \phi_{\mathrm{PO}}), \tag{1.15}$$

where L_{PO} is the length of a periodic orbit and the sum is over all such orbits. Define the "length spectrum" $R(L)$ as the Fourier transform of $\rho(k)$. Then, if the conjecture is correct it ought to display peaks at $L = L_j$. The periodic orbits in the circle are shown in Figure 1.3, parameterized by v, the number of vertices, and w, the winding number about the center. The length of each orbit is given by

$$L_{vw} = 2vR \sin \frac{\pi w}{v}, \tag{1.16}$$

where R is the radius of the circle, taken to be unity henceforth.

Because the eigenwavenumbers of the circular membrane are zeros of Bessel functions, which can be found numerically, the length spectrum can be easily calculated as

$$R(L) = \int_{-\infty}^{\infty} \sum_{m,n} \delta(k - \mathrm{j}_{mn}) \mathrm{e}^{\mathrm{i}kL} \, \mathrm{d}k = \sum_{m,n} \mathrm{e}^{\mathrm{i}\,\mathrm{j}_{mn} L}. \tag{1.17}$$

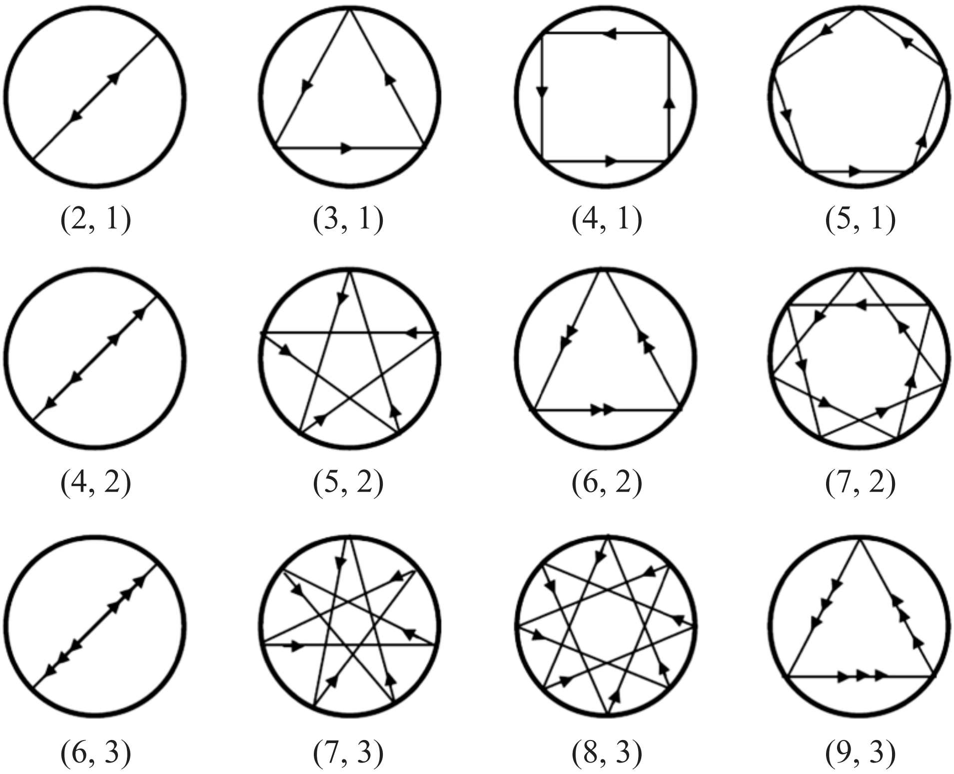

Figure 1.3. Periodic orbits for a circular domain. After Balian and Bloch (1972).

The absolute value of this is plotted in Figure 1.4. As expected from the preceding conjecture, it shows peaks at values of L satisfying Equation (1.16) for integer v and w, that is, 4, $3\sqrt{3}$, $4\sqrt{2}$, $10 \sin \pi/5$, and so on.

With this evidence we are ready to sketch the derivation of a formula like Equation (1.15) for any shape of membrane or cavity. First, however, we will find it helpful to review the quantum theory that gave rise to this result, and the analogy between quantum billiards and acoustical systems.

1.3 The Quantum–Acoustic Analogy

A widely studied problem in quantum physics is that of a scalar particle in a potential field, which obeys Schrödinger's equation:

$$\frac{-\hbar^2}{2m} \nabla^2 \psi_n + V(\mathbf{r})\psi_n = E_n \psi_n, \tag{1.18}$$

where $2\pi\hbar = 6.6 \times 10^{-34}$ Js is Planck's constant, m is the particle's mass, V is the potential at a point $\mathbf{r}$, and E_n is the nth discrete energy level. The complex wavefunction ψ_n can then be interpreted so that $|\psi_n(\mathbf{r})|^2 \, \mathrm{d}\mathbf{r}$ is the probability of finding a particle with energy E_n in the volume $\mathrm{d}\mathbf{r}$ surrounding the point $\mathbf{r}$. If the potential takes the form of an infinite well, so that it is zero within a domain B and infinite outside it, then the boundary condition will be $\psi_n = 0$ on ∂B, and the wavefunctions will be normalized such that $\int_B |\psi_n(\mathbf{r})|^2 \, \mathrm{d}\mathbf{r} = 1$ because the particle must exist