A RADICAL APPROACH TO LEBESGUE'S THEORY OF INTEGRATION

Meant for advanced undergraduate and graduate students in mathematics, this lively introduction to measure theory and Lebesgue integration is rooted in and motivated by the historical questions that led to its development. The author stresses the original purpose of the definitions and theorems and highlights some of the difficulties that were encountered as these ideas were refined.

The story begins with Riemann's definition of the integral, a definition created so that he could understand how broadly one could define a function and yet have it be integrable. The reader then follows the efforts of many mathematicians who wrestled with the difficulties inherent in the Riemann integral, leading to the work in the late nineteenth and early twentieth centuries of Jordan, Borel, and Lebesgue, who finally broke with Riemann's definition. Ushering in a new way of understanding integration, they opened the door to fresh and productive approaches to many of the previously intractable problems of analysis.

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi

Cambridge University Press
32 Avenue of the Americas, New York, NY 10013-2473, USA

www.cambridge.org
Information on this title: www.cambridge.org/9780521884747

The Mathematical Association of America
1529 Eighteenth Street, NW, Washington, DC 20036

First published 2008

Printed in the United States of America

A catalog record for this publication is available from the British Library.

Library of Congress Cataloging in Publication Data

Bressoud, David M., 1950–
A Radical approach to Lebesgue's theory of integration / David M. Bressoud.
p. cm. – (Classroom resource materials)
Includes bibliographical references and index.
ISBN-13: 978-0-521-88474-7 (hardback)
ISBN-10: 0-521-88474-8 (hardback)
ISBN-13: 978-0-521-71183-8 (pbk.)
ISBN-10: 0-521-71183-5 (pbk.)
1. Integrals, Generalized. I. Title. II. Series.
QA312.B67 2008
515′.42–dc22 2007035326

ISBN 978-0-521-88474-7 hardback
ISBN 978-0-521-71183-8 paperback

Dedicated to Herodotus,
the little lion of Cambridge Street,
and to the woman who loves him

Cambridge University Press
978-0-521-88474-7 - A Radical Approach to Lebesgue's Theory of Integration
David M. Bressoud
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Preface

I look at the burning question of the foundations of infinitesimal analysis without sorrow, anger, or irritation. What Weierstrass – Cantor – did was *very good. That's the way it* had *to be done. But whether this corresponds to what is in the depths of our consciousness is a very different question. I cannot but see a stark contradiction between the intuitively clear fundamental formulas of the integral calculus and the incomparably artificial and complex work of the "justification" and their "proofs." One must be quite stupid not to see this at once, and quite careless if, after having seen this, one can get used to this artificial, logical atmosphere, and can later on forget this stark contradiction.*

– Nikolaĭ Nikolaevich Luzin

Nikolaĭ Luzin reminds us of a truth too often forgotten in the teaching of analysis; the ideas, methods, definitions, and theorems of this study are neither natural nor intuitive. It is all too common for students to emerge from this study with little sense of how the concepts and results that constitute modern analysis hang together. Here more than anywhere else in the advanced undergraduate/beginning graduate curriculum, the historical context is critical to developing an understanding of the mathematics.

This historical context is both interesting and pedagogically informative. From transfinite numbers to the Heine–Borel theorem to Lebesgue measure, these ideas arose from practical problems but were greeted with a skepticism that betrayed confusion. Understanding what they mean and how they can be used was an uncertain process. We should expect our students to encounter difficulties at precisely those points at which the contemporaries of Weierstrass, Cantor, and Lebesgue had balked.

Throughout this text I have tried to emphasize that no one set out to invent measure theory or functional analysis. I find it both surprising and immensely satisfying that the search for understanding of Fourier series continued to be one of the principal driving forces behind the development of analysis well into the twentieth century. The tools that these mathematicians had at hand were not adequate to the task. In particular, the Riemann integral was poorly adapted to their needs.

It took several decades of wrestling with frustrating difficulties before mathematicians were willing to abandon the Riemann integral. The route to its eventual replacement, the Lebesgue integral, led through a sequence of remarkable insights into the complexities of the real number line. By the end of 1890s, it was recognized that analysis and the study of sets were inextricably linked. From this rich interplay, measure theory would emerge. With it came what today we call Lebesgue's dominated convergence theorem, the holy grail of nineteenth-century analysis. What so many had struggled so hard to discover now appeared as a gift that was almost free.

This text is an introduction to measure theory and Lebesgue integration, though anyone using it to support such a course must be forewarned that I have intentionally avoided stating results in their greatest possible generality. Almost all results are given only for the real number line. Theorems that are true over any compact set are often stated only for closed, bounded intervals. I want students to get a feel for these results, what they say, and why they are important. Close examination of the most general conditions under which conclusions will hold is something that can come later, if and when it is needed.

The title of this book was chosen to communicate two important points. First, this is a sequel to *A Radical Approach to Real Analysis* (ARATRA). That book ended with Riemann's definition of the integral. That is where this text begins. All of the topics that one might expect to find in an undergraduate analysis book that were not in ARATRA are contained here, including the topology of the real number line, fundamentals of set theory, transfinite cardinals, the Bolzano–Weierstrass theorem, and the Heine–Borel theorem. I did not include them in the first volume because I felt I could not do them justice there and because, historically, they are quite sophisticated insights that did not arise until the second half of the nineteenth century.

Second, this book owes a tremendous debt to Thomas Hawkins' *Lebesgue's Theory of Integration: Its Origins and Development*. Like ARATRA, this book is not intended to be read as a history of the development of analysis. Rather, this is a textbook informed by history, attempting to communicate the motivations, uncertainties, and difficulties surrounding the key concepts. This task would have been far more difficult without Hawkins as a guide. Those who are intrigued by the historical details encountered in this book are encouraged to turn to Hawkins and other historians of this period for fuller explanation.

Even more than ARATRA, this is the story of many contributions by many members of a large community of mathematicians working on different pieces of the puzzle. I hope that I have succeeded in opening a small window into the workings of this community. One of the most intriguing of these mathematicians is Axel

Harnack, who keeps reappearing in our story because he kept making mistakes, but they were *good* mistakes. Harnack's errors condensed and made explicit many of the misconceptions of his time, and so helped others to find the correct path. For ARATRA, it was easy to select the four mathematicians who should grace the cover: Fourier, Cauchy, Abel, and Dirichlet stand out as those who shaped the origins of modern analysis. For this book, the choice is far less clear. Certainly I need to include Riemann and Lebesgue, for they initiate and bring to conclusion the principal elements of this story. Weierstrass? He trained and inspired the generation that would grapple with Riemann's work, but his contributions are less direct. Heine, du Bois-Reymond, Jordan, Hankel, Darboux, or Dini? They all made substantial progress toward the ultimate solution, but none of them stands out sufficiently. Cantor? Certainly yes. It was his recognition that set theory lies at the heart of analysis that would enable the progress of the next generation. Who should we select from that next generation: Peano, Volterra, Borel, Baire? Maybe Riesz or one of the others who built on Lebesgue's insights, bringing them to fruition? Now the choice is even less clear. I have settled on Borel for his impact as a young mathematician and to honor him as the true source of the Heine–Borel theorem, a result that I have been very tempted to refer to as he did: the first fundamental theorem of measure theory.

I have drawn freely on the scholarship of others. I must pay special tribute to Soo Bong Chae's *Lebesgue Integration*. When I first saw this book, my reaction was that I did not need to write my own on Lebesgue integration. Here was someone who had already put the subject into historical context, writing in an elegant yet accessible style. However, as I have used his book over the years, I have found that there is much that he leaves unsaid, and I disagree with his choice to use Riesz's approach to the Lebesgue integral, building it via an analysis of step functions. Riesz found an elegant route to Lebesgue integration, but in defining the integral first and using it to define Lebesgue measure, the motivation for developing these concepts is lost. Despite such fundamental divergences, the attentive reader will discover many close parallels between Chae's treatment and mine.

I am indebted to many people who read and commented on early drafts of this book. I especially thank Dave Renfro who gave generously of his time to correct many of my historical and mathematical errors. Steve Greenfield had the temerity to be the very first reader of my very first draft, and I appreciate his many helpful suggestions on the organization and presentation of this book. I also want to single out my students who, during the spring semester of 2007, struggled through a preliminary draft of this book and helped me in many ways to correct errors and improve the presentation of this material. They are Jacob Bond, Kyle Braam, Pawan Dhir, Elizabeth Gillaspy, Dan Gusset, Sam Handler, Kassa Haileyesus, Xi Luo, Jake Norton, Stella Stamenova, and Linh To.

I am also grateful to the mathematicians and historians of mathematics who suggested corrections and changes or helped me find information. These include Roger Cooke, Larry D'Antonio, Ivor Grattan-Guinness, Daesuk Han, Tom Hawkins, Mark Huibregtse, Nicholas Rose, Peter Ross, Jim Smoak, John Stillwell, and Sergio B. Volchan. I am also indebted to Don Albers of the MAA and Lauren Cowles of Cambridge who so enthusiastically embraced this project, and to the reviewers for both MAA and Cambridge whose names are unknown to me but who gave much good advice.

Corrections, commentary, and additional material for this book can be found at `www.macalester.edu/aratra`.

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June 19, 2007