Chapter I

INTRODUCTION

I.1. The Main Question

The purpose of this book is to use the tools of mathematical logic to study certain problems in foundations of mathematics. We are especially interested in the question of which set existence axioms are needed to prove the known theorems of mathematics.

The scope of this initial question is very broad, but we can narrow it down somewhat by dividing mathematics into two parts. On the one hand there is set-theoretic mathematics, and on the other hand there is what we call "non-set-theoretic" or "ordinary" mathematics. By *set-theoretic mathematics* we mean those branches of mathematics that were created by the set-theoretic revolution which took place approximately a century ago. We have in mind such branches as general topology, abstract functional analysis, the study of uncountable discrete algebraic structures, and of course abstract set theory itself.

We identify as *ordinary* or *non-set-theoretic* that body of mathematics which is prior to or independent of the introduction of abstract set-theoretic concepts. We have in mind such branches as geometry, number theory, calculus, differential equations, real and complex analysis, countable algebra, the topology of complete separable metric spaces, mathematical logic, and computability theory.

The distinction between set-theoretic and ordinary mathematics corresponds roughly to the distinction between "uncountable mathematics" and "countable mathematics". This formulation is valid if we stipulate that "countable mathematics" includes the study of possibly uncountable complete separable metric spaces. (A metric space is said to be separable if it has a countable dense subset.) Thus for instance the study of continuous functions of a real variable is certainly part of ordinary mathematics, even though it involves an uncountable algebraic structure, namely the real number system. The point is that in ordinary mathematics, the real line partakes of countability since it is always viewed as a separable metric space, never as being endowed with the discrete topology.

In this book we want to restrict our attention to ordinary, non-set-theoretic mathematics. The reason for this restriction is that the set existence axioms which are needed for set-theoretic mathematics are likely to be much stronger than those which are needed for ordinary mathematics. Thus our broad set existence question really consists of two subquestions which have little to do with each other. Furthermore, while nobody doubts the importance of strong set existence axioms in set theory itself and in set-theoretic mathematics generally, the role of set existence axioms in ordinary mathematics is much more problematical and interesting.

We therefore formulate our *Main Question* as follows: *Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics*?

In any investigation of the Main Question, there arises the problem of choosing an appropriate language and appropriate set existence axioms. Since in ordinary mathematics the objects studied are almost always countable or separable, it would seem appropriate to consider a language in which countable objects occupy center stage. For this reason, we study the Main Question in the context of the language of second order arithmetic. This language is denoted L_2 and will be described in the next section. All of the set existence axioms which we consider in this book will be expressed as formulas of the language L_2.

I.2. Subsystems of Z_2

In this section we define Z_2, the formal system of second order arithmetic. We also introduce the concept of a subsystem of Z_2.

The *language of second order arithmetic* is a two-sorted language. This means that there are two distinct sorts of variables which are intended to range over two different kinds of object. Variables of the first sort are known as *number variables*, are denoted by $i, j, k, m, n, \dots$, and are intended to range over the set $\omega = \{0, 1, 2, \dots\}$ of all natural numbers. Variables of the second sort are known as *set variables*, are denoted by $X, Y, Z, \dots$, and are intended to range over all subsets of ω.

The terms and formulas of the language of second order arithmetic are as follows. *Numerical terms* are number variables, the constant symbols 0 and 1, and $t_1 + t_2$ and $t_1 \cdot t_2$ whenever t_1 and t_2 are numerical terms. Here $+$ and $\cdot$ are binary operation symbols intended to denote addition and multiplication of natural numbers. (Numerical terms are intended to denote natural numbers.) *Atomic formulas* are $t_1 = t_2$, $t_1 < t_2$, and $t_1 \in X$ where t_1 and t_2 are numerical terms and X is any set variable. (The intended meanings of these respective atomic formulas are that t_1 equals t_2, t_1 is less than t_2, and t_1 is an element of X.) *Formulas* are built up from atomic formulas by means of propositional connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$,

$\leftrightarrow$ (and, or, not, implies, if and only if), *number quantifiers* $\forall n$, $\exists n$ (for all n, there exists n), and *set quantifiers* $\forall X$, $\exists X$ (for all X, there exists X). A *sentence* is a formula with no free variables.

DEFINITION I.2.1 (language of second order arithmetic). L_2 is defined to be the language of second order arithmetic as described above.

In writing terms and formulas of L_2, we shall use parentheses and brackets to indicate grouping, as is customary in mathematical logic textbooks. We shall also use some obvious abbreviations. For instance, $2+2=4$ stands for $(1+1)+(1+1)=((1+1)+1)+1$, $(m+n)^2 \notin X$ stands for $\neg((m+n)\cdot(m+n)\in X)$, $s \leq t$ stands for $s<t \vee s=t$, and $\varphi \wedge \psi \wedge \theta$ stands for $(\varphi \wedge \psi) \wedge \theta$.

The semantics of the language L_2 are given by the following definition.

DEFINITION I.2.2 (L_2-structures). A *model for* L_2, also called a *structure for* L_2 or an L_2-*structure*, is an ordered 7-tuple

$$M = (|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M),$$

where $|M|$ is a set which serves as the range of the number variables, $\mathcal{S}_M$ is a set of subsets of $|M|$ serving as the range of the set variables, $+_M$ and $\cdot_M$ are binary operations on $|M|$, 0_M and 1_M are distinguished elements of $|M|$, and $<_M$ is a binary relation on $|M|$. We always assume that the sets $|M|$ and $\mathcal{S}_M$ are disjoint and nonempty. Formulas of L_2 are interpreted in M in the obvious way.

In discussing a particular model M as above, it is useful to consider formulas with parameters from $|M| \cup \mathcal{S}_M$. We make the following slightly more general definition.

DEFINITION I.2.3 (parameters). Let $\mathcal{B}$ be any subset of $|M| \cup \mathcal{S}_M$. By a *formula with parameters from* $\mathcal{B}$ we mean a formula of the extended language $\mathrm{L}_2(\mathcal{B})$. Here $\mathrm{L}_2(\mathcal{B})$ consists of L_2 augmented by new constant symbols corresponding to the elements of $\mathcal{B}$. By a *sentence with parameters from* $\mathcal{B}$ we mean a sentence of $\mathrm{L}_2(\mathcal{B})$, i.e., a formula of $\mathrm{L}_2(\mathcal{B})$ which has no free variables.

In the language $\mathrm{L}_2(|M| \cup \mathcal{S}_M)$, constant symbols corresponding to elements of $\mathcal{S}_M$ (respectively $|M|$) are treated syntactically as unquantified set variables (respectively unquantified number variables). Sentences and formulas with parameters from $|M| \cup \mathcal{S}_M$ are interpreted in M in the obvious way. A set $A \subseteq |M|$ is said to be *definable over* M *allowing parameters from* $\mathcal{B}$ if there exists a formula $\varphi(n)$ with parameters from $\mathcal{B}$ and no free variables other than n such that

$$A = \{a \in |M| \colon M \models \varphi(a)\}.$$

Here $M \models \varphi(a)$ means that M *satisfies* $\varphi(a)$, i.e., $\varphi(a)$ is true in M.

We now discuss some specific L_2-structures. The *intended model* for L_2 is of course the model

$$(\omega, P(\omega), +, \cdot, 0, 1, <)$$

where ω is the set of natural numbers, $P(\omega)$ is the set of all subsets of ω, and $+, \cdot, 0, 1, <$ are as usual. By an *ω-model* we mean an L_2-structure of the form

$$(\omega, \mathcal{S}, +, \cdot, 0, 1, <)$$

where $\emptyset \neq \mathcal{S} \subseteq P(\omega)$. Thus an ω-model differs from the intended model only by having a possibly smaller collection $\mathcal{S}$ of sets to serve as the range of the set variables. We sometimes speak of the *ω-model $\mathcal{S}$* when we really mean the ω-model $(\omega, \mathcal{S}, +, \cdot, 0, 1, <)$. In some parts of this book we shall be concerned with a special class of ω-models known as β-models. This class will be defined in §I.5.

We now present the formal system of second order arithmetic.

DEFINITION I.2.4 (second order arithmetic). The *axioms of second order arithmetic* consist of the universal closures of the following L_2-formulas:

(i) basic axioms:

$$\begin{array}{l} n+1 \neq 0 \\ m+1 = n+1 \rightarrow m = n \\ m+0 = m \\ m+(n+1) = (m+n)+1 \\ m \cdot 0 = 0 \\ m \cdot (n+1) = (m \cdot n)+m \\ \neg m < 0 \\ m < n+1 \leftrightarrow (m < n \vee m = n) \end{array}$$

(ii) induction axiom:

$$(0 \in X \wedge \forall n\,(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n\,(n \in X)$$

(iii) comprehension scheme:

$$\exists X\, \forall n\, (n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any formula of L_2 in which X does not occur freely.

Intuitively, the given instance of the comprehension scheme says that there exists a set $X = \{n : \varphi(n)\}$ = the set of all n such that $\varphi(n)$ holds. This set is said to be *defined by* the given formula $\varphi(n)$. For example, if $\varphi(n)$ is the formula $\exists m\,(m+m=n)$, then this instance of the comprehension scheme asserts the existence of the set of even numbers.

In the comprehension scheme, $\varphi(n)$ may contain free variables in addition to n. These free variables may be referred to as *parameters* of this instance of the comprehension scheme. Such terminology is in harmony with definition I.2.3 and the discussion following it. For example, taking $\varphi(n)$ to be the formula $n \notin Y$, we have an instance of comprehension,

$$\forall Y\, \exists X\, \forall n\, (n \in X \leftrightarrow n \notin Y),$$

asserting that for any given set Y there exists a set $X =$ the complement of Y. Here the variable Y plays the role of a parameter.

Note that an L_2-structure M satisfies I.2.4(iii), the comprehension scheme, if and only if $\mathcal{S}_M$ contains all subsets of $|M|$ which are definable over M allowing parameters from $|M| \cup \mathcal{S}_M$. In particular, the comprehension scheme is valid in the intended model. Note also that the basic axioms I.2.4(i) and the induction axiom I.2.4(ii) are valid in any ω-model. In fact, any ω-model satisfies the full *second order induction scheme*, i.e., the universal closure of

$$(\varphi(0) \wedge \forall n\, (\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n\, \varphi(n),$$

where $\varphi(n)$ is any formula of L_2. In addition, the second order induction scheme is valid in any model of I.2.4(ii) plus I.2.4(iii).

By *second order arithmetic* we mean the formal system in the language L_2 consisting of the axioms of second order arithmetic, together with all formulas of L_2 which are deducible from those axioms by means of the usual logical axioms and rules of inference. The formal system of second order arithmetic is also known as Z_2, for obvious reasons, or Π^1_∞-CA_0, for reasons which will become clear in §I.5.

In general, a *formal system* is defined by specifying a language and some axioms. Any formula of the given language which is logically deducible from the given axioms is said to be a *theorem* of the given formal system. At all times we assume the usual logical rules and axioms, including equality axioms and the law of the excluded middle.

This book will be largely concerned with certain specific subsystems of second order arithmetic and the formalization of ordinary mathematics within those systems. By a *subsystem of* Z_2 we mean of course a formal system in the language L_2 each of whose axioms is a theorem of Z_2. When introducing a new subsystem of Z_2, we shall specify the axioms of the system by writing down some formulas of L_2. The axioms are then taken to be the universal closures of those formulas.

If T is any subsystem of Z_2, a *model of* T is any L_2-structure satisfying the axioms of T. By Gödel's completeness theorem applied to the two-sorted language L_2, we have the following important principle: A given L_2-sentence σ is a theorem of T if and only if all models of T satisfy σ. An *ω-model of* T is of course any ω-model which satisfies the axioms of T, and similarly a *β-model of* T is any β-model satisfying the axioms of T. Chapters VII, VIII, and IX of this book constitute a thorough study of models of subsystems of Z_2. Chapter VII is concerned with β-models, chapter VIII is concerned with ω-models other than β-models, and chapter IX is concerned with models other than ω-models.

All of the subsystems of Z_2 which we shall consider consist of the basic axioms I.2.4(i), the induction axiom I.2.4(ii), and some set existence axioms. The various subsystems will differ from each other only with respect to their set existence axioms. Recall from §I.1 that our Main Question concerns the role of set existence axioms in ordinary mathematics. Thus, a principal theme of this book will be the formal development of specific portions of ordinary mathematics within specific subsystems of Z_2. We shall see that subsystems of Z_2 provide a setting in which the Main Question can be investigated in a precise and fruitful way. Although Z_2 has infinitely many subsystems, it will turn out that only a handful of them are useful in our study of the Main Question.

Notes for §I.2. The formal system Z_2 of second order arithmetic was introduced in Hilbert/Bernays [115] (in an equivalent form, using a somewhat different language and axioms). The development of a portion of ordinary mathematics within Z_2 is outlined in Supplement IV of Hilbert/Bernays [115]. The present book may be regarded as a continuation of the research begun by Hilbert and Bernays.

I.3. The System ACA_0

The previous section contained generalities about subsystems of Z_2. The purpose of this section is to introduce a particular subsystem of Z_2 which is of central importance, namely ACA_0.

In our designation ACA_0, the acronym ACA stands for arithmetical comprehension axiom. This is because ACA_0 contains axioms asserting the existence of any set which is arithmetically definable from given sets (in a sense to be made precise below). The subscript 0 denotes restricted induction. This means that ACA_0 does not include the full second order induction scheme (as defined in §I.2). We assume only the induction axiom I.2.4(ii).

We now proceed to the definition of ACA_0.

DEFINITION I.3.1 (arithmetical formulas). A formula of L_2, or more generally a formula of $L_2(|M| \cup \mathcal{S}_M)$ where M is any L_2-structure, is said to be *arithmetical* if it contains no set quantifiers, i.e., all of the quantifiers appearing in the formula are number quantifiers.

Note that arithmetical formulas of L_2 may contain free set variables, as well as free and bound number variables and number quantifiers. Arithmetical formulas of $L_2(|M| \cup \mathcal{S}_M)$ may additionally contain set parameters and number parameters, i.e., constant symbols denoting fixed elements of $\mathcal{S}_M$ and $|M|$ respectively.

Examples of arithmetical formulas of L_2 are

$$\forall n\,(n \in X \rightarrow \exists m\,(m + m = n)),$$

asserting that all elements of the set X are even, and

$$\forall m\,\forall k\,(n = m \cdot k \rightarrow (m = 1 \vee k = 1)) \wedge n > 1 \wedge n \in X,$$

asserting that n is a prime number and is an element of X. An example of a non-arithmetical formula is

$$\exists Y\,\forall n\,(n \in X \leftrightarrow \exists i\,\exists j\,(i \in Y \wedge j \in Y \wedge i + n = j))$$

asserting that X is the set of differences of elements of some set Y.

DEFINITION I.3.2 (arithmetical comprehension). The *arithmetical comprehension scheme* is the restriction of the comprehension scheme I.2.4(iii) to arithmetical formulas $\varphi(n)$. Thus we have the universal closure of

$$\exists X\,\forall n\,(n \in X \leftrightarrow \varphi(n))$$

whenever $\varphi(n)$ is a formula of L_2 which is arithmetical and in which X does not occur freely. ACA_0 is the subsystem of Z_2 whose axioms are the arithmetical comprehension scheme, the induction axiom I.2.4(ii), and the basic axioms I.2.4(i).

Note that an L_2-structure

$$M = (|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M)$$

satisfies the arithmetical comprehension scheme if and only if $\mathcal{S}_M$ contains all subsets of $|M|$ which are definable over M by arithmetical formulas with parameters from $|M| \cup \mathcal{S}_M$. Thus, a model of ACA_0 is any such L_2-structure which in addition satisfies the induction axiom and the basic axioms.

An easy consequence of the arithmetical comprehension scheme and the induction axiom is the *arithmetical induction scheme*:

$$(\varphi(0) \wedge \forall n\,(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n\,\varphi(n)$$

for all L_2-formulas $\varphi(n)$ which are arithmetical. Thus any model of ACA_0 is also a model of the arithmetical induction scheme. (Note however that ACA_0 does not include the second order induction scheme, as defined in §I.2.)

REMARK I.3.3 (first order arithmetic). We wish to remark that there is a close relationship between ACA_0 and first order arithmetic. Let L_1 be the *language of first order arithmetic*, i.e., L_1 is just L_2 with the set variables omitted. *First order arithmetic* is the formal system Z_1 whose language

Cambridge University Press
978-0-521-88439-6 - Subsystems of Second Order Arithmetic, Second Edition
Stephen G. Simpson
Excerpt
More information

is L_1 and whose axioms are the basic axioms I.2.4(i) plus the *first order induction scheme*:

$$(\varphi(0) \wedge \forall n\,(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n\,\varphi(n)$$

for all L_1-formulas $\varphi(n)$. In the literature of mathematical logic, first order arithmetic is sometimes known as *Peano arithmetic*, PA. By the previous paragraph, every theorem of Z_1 is a theorem of ACA_0. In model-theoretic terms, this means that for any model $(|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M)$ of ACA_0, its first order part $(|M|, +_M, \cdot_M, 0_M, 1_M, <_M)$ is a model of Z_1. In §IX.1 we shall prove a converse to this result: Given a model

$$(|M|, +_M, \cdot_M, 0_M, 1_M, <_M) \tag{1}$$

of first order arithmetic, we can find $\mathcal{S}_M \subseteq P(|M|)$ such that

$$(|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M)$$

is a model of ACA_0. (Namely, we can take $\mathcal{S}_M = \mathrm{Def}(M) =$ the set of all $A \subseteq |M|$ such that A is definable over (1) allowing parameters from $|M|$.) It follows that, for any L_1-sentence σ, σ is a theorem of ACA_0 if and only if σ is a theorem of Z_1. In other words, ACA_0 is a *conservative extension* of first order arithmetic. This may also be expressed by saying that Z_1, or equivalently PA, is the *first order part* of ACA_0. For details, see §IX.1.

REMARK I.3.4 (ω-models of ACA_0). Assuming familiarity with some basic concepts of recursive function theory, we can characterize the ω-models of ACA_0 as follows. $\mathcal{S} \subseteq P(\omega)$ is an ω-model of ACA_0 if and only if

(i) $\mathcal{S} \neq \emptyset$;
(ii) $A \in \mathcal{S}$ and $B \in \mathcal{S}$ imply $A \oplus B \in \mathcal{S}$;
(iii) $A \in \mathcal{S}$ and $B \leq_{\mathrm{T}} A$ imply $B \in \mathcal{S}$;
(iv) $A \in \mathcal{S}$ implies $\mathrm{TJ}(A) \in \mathcal{S}$.

(This result is proved in §VIII.1.)

Here $A \oplus B$ is the *recursive join* of A and B, defined by

$$A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}.$$

$B \leq_{\mathrm{T}} A$ means that B is *Turing reducible* to A, i.e., B is *recursive in* A, i.e., the characteristic function of B is computable assuming an oracle for the characteristic function of A. $\mathrm{TJ}(A)$ denotes the *Turing jump* of A, i.e., the complete recursively enumerable set relative to A.

In particular, ACA_0 has a minimum (i.e., unique smallest) ω-model, namely

$$\mathrm{ARITH} = \{A \in P(\omega) : \exists n \in \omega\,(A \leq_{\mathrm{T}} \mathrm{TJ}(n, \emptyset))\},$$

where $\mathrm{TJ}(n, X)$ is defined inductively by $\mathrm{TJ}(0, X) = X$, $\mathrm{TJ}(n+1, X) = \mathrm{TJ}(\mathrm{TJ}(n, X))$. More generally, given a set $B \in P(\omega)$, there is a unique smallest ω-model of ACA_0 containing B, consisting of all sets which are arithmetical in B. (For $A, B \in P(\omega)$, we say that A is *arithmetical in B* if $A \leq_T \mathrm{TJ}(n, B)$ for some $n \in \omega$. This is equivalent to saying that A is definable in some or any ω-model $(\omega, \mathcal{S}, +, \cdot, 0, 1, <)$, $B \in \mathcal{S} \subseteq P(\omega)$, by an arithmetical formula with B as a parameter.)

Models of ACA_0 are discussed further in §§VIII.1, IX.1, and IX.4. The development of ordinary mathematics within ACA_0 is discussed in §I.4 and in chapters II, III, and IV.

Notes for §I.3. By remark I.3.3, the system ACA_0 is closely related to first order arithmetic. First order arithmetic is one of the best known and most studied formal systems in the literature of mathematical logic. See for instance Hilbert/Bernays [115], Mendelson [185, chapter 3], Takeuti [261, chapter 2], Shoenfield [222, chapter 8], Hájek/Pudlák [100], and Kaye [137]. By remark I.3.4, ω-models of ACA_0 are closely related to basic concepts of recursion theory such as relative recursiveness, the Turing jump operator, and the arithmetical hierarchy. For an introduction to these concepts, see for instance Rogers [208, chapters 13–15], Shoenfield [222, chapter 7], Cutland [43], or Lerman [161, chapters I–III].

I.4. Mathematics within ACA_0

The formal system ACA_0 was introduced in the previous section. We now outline the development of certain portions of ordinary mathematics within ACA_0. The material presented in this section will be restated and greatly refined and extended in chapters II, III, and IV. The present discussion is intended as a partial preview of those chapters.

If X and Y are set variables, we use $X = Y$ and $X \subseteq Y$ as abbreviations for the formulas $\forall n\,(n \in X \leftrightarrow n \in Y)$ and $\forall n\,(n \in X \rightarrow n \in Y)$ respectively.

Within ACA_0, we define $\mathbb{N}$ to be the unique set X such that $\forall n\,(n \in X)$. (The existence of this set follows from arithmetical comprehension applied to the formula $\varphi(n) \equiv n = n$.) Thus, in any model

$$M = (|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M)$$

of ACA_0, $\mathbb{N}$ denotes $|M|$, the set of natural numbers in the sense of M, and we have $|M| \in \mathcal{S}_M$. We shall distinguish between $\mathbb{N}$ and ω, reserving ω to denote the set of natural numbers in the sense of “the real world,” i.e., the metatheory in which we are working, whatever that metatheory might be.

Within ACA_0, we define a *numerical pairing function* by

$$(m, n) = (m + n)^2 + m.$$

Within ACA_0 we can prove that, for all $m, n, i, j \in \mathbb{N}$, $(m, n) = (i, j)$ if and only if $m = i$ and $n = j$. Moreover, using arithmetical comprehension, we can prove that for all sets $X, Y \subseteq \mathbb{N}$, there exists a set $X \times Y \subseteq \mathbb{N}$ consisting of all (m, n) such that $m \in X$ and $n \in Y$. In particular we have $\mathbb{N} \times \mathbb{N} \subseteq \mathbb{N}$.

For $X, Y \subseteq \mathbb{N}$, a *function* $f : X \to Y$ is defined to be a set $f \subseteq X \times Y$ such that for all $m \in X$ there is exactly one $n \in Y$ such that $(m, n) \in f$. For $m \in X$, $f(m)$ is defined to be the unique n such that $(m, n) \in f$. The usual properties of such functions can be proved in ACA_0. In particular, we have *primitive recursion*. This means that, given $f : X \to Y$ and $g : \mathbb{N} \times X \times Y \to Y$, there is a unique $h : \mathbb{N} \times X \to Y$ defined by $h(0, m) = f(m)$, $h(n+1, m) = g(n, m, h(n, m))$ for all $n \in \mathbb{N}$ and $m \in X$. The existence of h is proved by arithmetical comprehension, and the uniqueness of h is proved by arithmetical induction. (For details, see §II.3.) In particular, we have the *exponential function* $\exp(m, n) = m^n$, defined by $m^0 = 1$, $m^{n+1} = m^n \cdot m$ for all $m, n \in \mathbb{N}$. The usual properties of the exponential function can be proved in ACA_0.

In developing ordinary mathematics within ACA_0, our first major task is to set up the *number systems*, i.e., the natural numbers, the integers, the rational number system, and the real number system.

The natural number system is essentially already given to us by the language and axioms of ACA_0. Thus, within ACA_0, a *natural number* is defined to be an element of $\mathbb{N}$, and the *natural number system* is defined to be the structure $\mathbb{N}, +_\mathbb{N}, \cdot_\mathbb{N}, 0_\mathbb{N}, 1_\mathbb{N}, <_\mathbb{N}, =_\mathbb{N}$, where $+_\mathbb{N} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by $m +_\mathbb{N} n = m + n$, etc. (Thus for instance $+_\mathbb{N}$ is the set of triples $((m, n), k) \in (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ such that $m + n = k$. The existence of this set follows from arithmetical comprehension.) This means that, when we are working within any particular model $M = (|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M)$ of ACA_0, a natural number is any element of $|M|$, and the role of the natural number system is played by $|M|, +_M, \cdot_M, 0_M, 1_M, <_M, =_M$. (Here $=_M$ is the identity relation on $|M|$.)

Basic properties of the natural number system, such as uniqueness of prime power decomposition, can be proved in ACA_0 using arithmetical induction. (Here one can follow the usual development within first order arithmetic, as presented in textbooks of mathematical logic. Alternatively, see chapter II.)

In order to define the set $\mathbb{Z}$ of *integers* within (any model of) ACA_0, we first use arithmetical comprehension to prove the existence of an equivalence relation $\equiv_\mathbb{Z} \subseteq (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$ defined by $(m, n) \equiv_\mathbb{Z} (i, j)$ if and only if $m + j = n + i$. We then use arithmetical comprehension again, this time with $\equiv_\mathbb{Z}$ as a parameter, to prove the existence of the set $\mathbb{Z}$ consisting of all $(m, n) \in \mathbb{N} \times \mathbb{N}$ such that that (m, n) is the minimum element of its equivalence class with respect to $\equiv_\mathbb{Z}$. (Here minimality is taken with respect to $<_\mathbb{N}$, using the fact that $\mathbb{N} \times \mathbb{N}$ is a subset of $\mathbb{N}$. Thus $\mathbb{Z}$ consists of one element of each $\equiv_\mathbb{Z}$-equivalence class.) Define $+_\mathbb{Z} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by letting $(m, n) +_\mathbb{Z} (i, j)$ be the unique element