ABSTRACT ELEMENTARY CLASSES: SOME ANSWERS, MORE QUESTIONS

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Abstract. We survey some of the recent work in the study of Abstract Elementary Classes focusing on the categoricity spectrum and the introduction of certain conditions (amalgamation, tameness, arbitrarily large models) which allow one to develop a workable theory. We repeat or raise for the first time a number of questions; many now seem to be accessible.

Much late 19th and early 20th century work in logic was in a 2nd order framework; infinitary logics in the modern sense were foreshadowed by Schroeder and Pierce before being formalized in modern terms in Poland during the late 20's. First order logic was only singled out as the 'natural' language to formalize mathematics as such authors as Tarski, Robinson, and Malcev developed the fundamental tools and applied model theory in the study of algebra. Serious work extending the model theory of the 50's to various infinitary logics blossomed during the 1960's and 70's with substantial work on logics such as $L_{\omega_1,\omega}$ and $L_{\omega_1,\omega}(Q)$. At the same time Shelah's work on stable theories completed the switch in focus in first order model theory from study of the logic to the study of complete first order theories As Shelah in [44, 46] sought to bring this same classification theory standpoint to infinitary logic, he introduced a total switch to a semantic standpoint. Instead of studying theories in a logic, one studies the class of models defined by a theory. He abstracted (pardon the pun) the essential features of the class of models of a first order theory partially ordered by the elementary submodel relation. An abstract elementary class AEC $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ is a class of models closed under isomorphism and partially ordered under $\prec_{\boldsymbol{K}}$, where $\prec_{\boldsymbol{K}}$ is required to refine the substructure relation, that is closed under unions and satisfies two additional conditions: if each element M_i of a chain satisfies $M_i \prec_{\boldsymbol{K}} M$ then $M_0 \prec_{\boldsymbol{K}} \bigcup_i M_i \prec_{\boldsymbol{K}} M$ and $M_0 \prec_{\boldsymbol{K}} M_2, M_1 \prec_{\boldsymbol{K}} M_2$ and $M_0 \subseteq M_1$ implies $M_0 \prec_{\boldsymbol{K}} M_1$ (coherence axiom). Further there is a Löwnenheim-Skolem number κ associated with $\boldsymbol{K}$ so that if $A \subseteq M \in \boldsymbol{K}$, there is an M_1 with $A \subset M_1 \prec_{\boldsymbol{K}} M$ and $|M_1| \leq |A| + \kappa$.

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In this paper we will review some of the reasons for considering AEC's, outline several major lines of study in the subject, and offer a series of problems whose solution would advance the various lines. The fundamental ideas discussed here are due to Shelah. However, we explore in some detail areas that have been developed in the very recent past by such authors as Grossberg, Hyttinen, Kolesnikov, Lessmann, VanDieren, and Villaveces; generally speaking these studies proceed by putting further model theoretic conditions on an AEC and we will expound some of these conditions. In the closing pages we give a short introduction to the mainline of Shelah's research [52, 53, 51, 50].

Our survey focuses primarily on problems closely related to categoricity. We have attempted to attribute both results and questions correctly. But many of the questions are just writing out what people in the area are thinking about. For expositional purposes, we frequently cite [1]; the default is that results in that monograph are not new although the proofs may be. I thank Tapani Hyttinen, the anonymous referee, and especially Rami Grossberg for useful comments on this article.

Increased interest in nonelementary classes arose recently for several reasons. First, the increased emphasis, signaled in [49, 54] and emphasized in [21], on hypotheses such as amalgamation or tameness as fruitful conditions to create a workable theory of AEC, has led to a number of new results. The need for studying AEC became more clear for two reasons. On the one hand the pursuit of specific problems in the first order setting has led to constructions which can no longer be formalized by first order means. On the other, the paradigm: study an interesting structure by studying its first order theory has broken down in some significant cases because the first order theory is not sufficiently nice.

The work of Kim and Pillay [31] showed that the essential distinction between stable and simple theories [45] lay in the fact that for a stable theory, Lascar strong type equals strong type. Strong types are first order objects; Lascar strong types are not. Analysis of this problem led to the introduction of hyperimaginaries and other properly infinitary objects and ultimately to compact abstract theories CATS [12]. In a slightly different direction, the 'Hrushovski construction' [28, 27] leads in nice cases (when the generic is ω-saturated) to the construction of first order theories with special properties. However, in certain notable cases, the best that has so far been found is a Robinson theory (in the search for a bad field [3, 5]) or even only a positive Robinson theory (in the search for a simple theory where strong type is not equal to Lascar strong type [40]). Despite the terminology, a (positive) Robinson theory, refers to the class of models of a first order theory which omit certain types; it can be described only in infinitary logic.

The first order theory of the field of complex numbers with exponentiation is intractable; the ring of integers and their order is first order definable. But Zilber suggested in a sequence of papers [60, 59, 57, 58] the notion of considering the $L_{\omega_1,\omega}(Q)$-theory of $(\mathbb{C}, +, \cdot, \exp)$. The intuition is that the essential wildness will be contained by forcing the kernel of the exponential map to always be exactly the standard integers. In his proof of categoricity for quasiminimal excellent classes Zilber discovered a special case of Shelah's notion of excellence that is easy to describe. He works in a context where there is a well-behaved notion of closure, cl which defines a combinatorial geometry. The aim is to show that if X is isomorphic to Y, then $\mathrm{cl}(X)$ is isomorphic to $\mathrm{cl}(Y)$. In general, this condition is non-trivial; it follows from excellence. In this context, excellence means that for every n, if $A = \{a_1, \ldots, a_n\}$ is an independent set then for any $a \in \mathrm{cl}(A)$ the type of a over $Z = \bigcup_{i<n} \mathrm{cl}(A - \{a_i\})$ is determined by the type of a over a finite subset of Z. Shelah works in a more general situation, where combinatorial geometry is replaced by a 'forking'-like notion. Consequently his notion is harder to describe and we omit the description here. Crucially, in both cases a condition (excellence) on countable models has important consequences (e.g. amalgamation) in all cardinalities.

Various other attempts to formalize analytic structures (notably Banach spaces [25, 26]) provide examples of 'homogeneous model theory' ([43, 13] and many more); Banach spaces are also an example of CATS [11]. Strictly speaking, the class of Banach spaces is not closed under unions of chains so doesn't form an AEC. But, Banach space model theory can be thought of as the study not of Banach spaces, but of structures whose completion is a Banach Space and this provides an interpretation of classes of Banach spaces as AEC's. Further mathematical examples include locally finite groups [19] and some aspects of compact complex manifolds (Although here, the first order theory is an attractive topic for model theorists (e.g. [39, 41]).).

Many, but not all, of these 'infinitary' formalizations can be captured in the framework of AEC's. (In particular, CATS are inherently different.) The work that I'll describe here has a complementary motivation. Stability theory provided a very strong tool to classify first order theories and then for extremely well-behaved theories (those below the 'main gap') to assign invariants to models of the theories. This insight of Shelah spread beyond stable theories with the realization that very different tools but some of the same heuristics allowed the study of o-minimal theories. By these techniques, o-minimality and stability, model theorists have learned much about the theories of both the real and the complex numbers and many other algebraic structures. But Shelah asks an in some ways more basic question. What are the properties of first order logic that make stability theory work? To what extent can we extend our results to wider classes, in particular to AEC?

Most known mathematical results are either *extremely cardinal dependent:* about finite or countable structures or at most structures of cardinality the continuum; *or completely cardinal independent:* about every structure satisfying certain properties. Already first order model theory has discovered problems that have an intimate relation between the cardinality of structures

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and algebraic properties of the structures:

1. Stability spectrum and counting models
2. A general theory of independence
3. Decomposition theorems for general models

There are structural algebraic, not merely combinatorial features, which are non-trivially cardinal dependent. (For example, the general theory of independence is intimately related with the class of cardinals in which the theory is stable and even for stable first order theories, stability in κ depends on the cofinality of κ.)

As usual a class of models $\boldsymbol{K}$ with a distinguished notion of submodel has *joint embedding property* (jep) if any two members of $\boldsymbol{K}$ have a common extension and $\boldsymbol{K}$ has the *amalgamation property* (ap) if any two extensions of a fixed model M have a common extension (over M).

If we were to take the fundamental analogy to be that an abstract elementary class represents a *complete* first order theory then we would add to the definition that the class $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ has the amalgamation and the joint embedding property. But completeness is a bit much to ask even in $L_{\omega_1,\omega}$. Here completeness (all models Karp equivalent) is not necessarily compatible with Löwenheim number ω. Some uncountable models do not have countable Karp equivalent submodels. The standard first order proof of the theorem, 'categoricity in power implies completeness' is a triviality but it assumes *both* the upwards and downwards Löwenheim-Skolem theorem for a set of sentences. Even for a sentence of $L_{\omega_1,\omega}$ in a countable language the reduction for an arbitrary categorical sentence ψ to one which is complete and has essentially the same spectrum is not at all trivial [44, 46]. It is substantially easier if ψ is assumed to have arbitrarily large models ([1] VII.2) than without that hypothesis ([1] VII.3). The difficult case is carried out in full in [44, 46]; the easier case is hinted at in [44, 46] but spelled out in the expository [3, 1]. In either case a notion of stability (counting the number of types) is used to obtain even the completeness result.

Moreover, unlike the first order case, completeness does not immediately yield the amalgamation property. The only known proof [46, 47] that a categorical sentence in $L_{\omega_1,\omega}$ has the amalgamation property invokes the weak continuum hypothesis and introduces the much more intricate notion of excellence. Moreover few models in every cardinal up to $\aleph_\omega$ is assumed; indeed, categoricity in every cardinal up to $\aleph_\omega$ is essential to get eventual categoricity [24, 1]. Similarly, although Zilber's quasiminimal excellent classes do have the amalgamation property the existing proof deduces the result from the proof of excellence, which has non-trivial algebraic content (e.g. [57]).

We will discuss first AEC with arbitrarily large models and then move to a harder case where that assumption is not made.

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QUESTION 1. Must the class of models of a sentence in $L_{\omega_1,\omega}$ (or more generally an AEC) that has arbitrarily large models and is categorical in a sufficiently large cardinal have the amalgamation property (at least for sufficiently large models). This is an interesting question even assuming the weak GCH; the necessity of such an assumption presents a different set of problems.

Grossberg (e.g. [15]) has posed this question for AEC and for $L_{\omega_1,\omega}$. For sentences of $L_{\omega_1,\omega}$, Shelah's result reported above gives a partial answer modulo weak gch. He deduces excellence and thus amalgamation from categoricity up to $\aleph_\omega$. But although Grossberg's question is on the 'assume arbitrarily large models side', it is more demanding than Shelah's result in asking that categoricity in one cardinality suffice. Trying to obtain a proof (even for $L_{\omega_1,\omega}$) from the arbitrarily large model assumption without passing through excellence is a 'warm-up' strategy for the AEC version.

Shelah's presentation theorem is a crucial tool for the study of AEC. It asserts that every AEC $\boldsymbol{K}$ may be seen as the class of reducts of a collection of models defined by a first order theory (in a language of size $\mathrm{LS}(\boldsymbol{K})$) which omit a specified collection of (at most $2^{\mathrm{LS}(\boldsymbol{K})}$) types. Let us state a crucial corollary. Fix a vocabulary τ. For each pair of a first order theory and set of types Φ (in a vocabulary τ' extending τ), and each linear order I, $EM(I, \Phi)$ denotes the reduct to τ of the τ'-structure which satisfies Φ. The presentation theorem implies that for each $\boldsymbol{K}$, there is a Φ such that $EM(\_, \Phi)$ is a functor into $\boldsymbol{K}$ (which takes subordering to $\prec_{\boldsymbol{K}}$). A straightforward use of Ehrenfeucht-Mostowski models over indiscernibles yields: If $\boldsymbol{K}$ has a model of cardinality greater than $\beth_{(2^{\mathrm{LS}(\boldsymbol{K})})^+}$ then $\boldsymbol{K}$ has arbitrarily large models. In the vernacular, we say the Hanf number for AEC with vocabulary of size at most κ and Löwenheim-Skolem number at most κ is at most $H(\kappa) = \beth_{(2^\kappa)^+}$. We call this function H as we use it to compute Hanf numbers. It might be more appropriate to call it ER as it actually computes the bound for applying the Erdos-Rado theorem to obtain indiscernibles.

Many of the ideas expounded here were presaged in earlier work such as [37, 33] dealing with languages $L_{\kappa,\omega}$ with strong hypotheses (e.g. compactness, measurability) on the cardinal κ. The earliest result in this series was:

THEOREM 2. [37] *Let κ be strongly compact. If a sentence $\psi \in L_{\kappa,\omega}$ is categorical in $\lambda^+ > \mu_0 = \beth_{(2^\kappa)^+}$ then it is categorical in all cardinals greater than μ_0.*

In view of the set-theoretic requirements on the syntax of the underlying logic, we don't discuss this line but deal with the more general notion of AEC.

For most of the rest of this paper, we will assume $\boldsymbol{K}$ is an AEC with the amalgamation property. It is then trivial to reduce to the study of AEC with both the amalgamation and joint embedding properties. Under these hypotheses, when $\boldsymbol{K}$ has arbitrarily large models, we are able to work inside a monster model which behaves much like the first order situation but is weaker in a significant way. We have amalgamation only over submodels, thus the monster model is homogeneous only over submodels. The stronger condition, assuming that there is a 'monster model' that is homogeneous over sets, gives rise to the area known as homogeneous model theory. For the major literature in this area consult such authors as Hyttinen, Lessmann, and Shelah.

Working within a model-homogeneous 'monster model' (i.e. in an AEC with amalgamation), we define the Galois type of a over M to be the orbit of a under automorphisms of the monster which fix M. We write $\mathcal{S}(M)$ for the collection of Galois types over M. Then we can define a model M to be κ-saturated if every Galois type over a submodel of M with cardinality $< \kappa$ is realized in M. A somewhat more general definition (without assuming ap) occurs in [48, 49].

We begin by discussing classes which have arbitrarily large models. Invoking the presentation theorem, we are able to build Ehrenfeucht-Mostowski models over sequences of order indiscernibles. As Shelah remarks in the introduction to [52], this yields the non-definability of well-ordering and so gives us an approximation to compactness. Most of these notes concern this case and build on [49]. We return at the end to the much more difficult situation, where one attempts to find information about AEC simply from the information that it has one (or few models) in some specific cardinalities. We will sketch some of Shelah's extensive work on this subject; our emphasis on classes with arbitrarily large models represents the extent of our understanding, not importance.

Assuming $\boldsymbol{K}$ has arbitrarily large models, the proof that categoricity in λ implies stability in all cardinals smaller than λ has the same general form as in the first order case. But other arguments involving Galois types over models generated by order indiscernibles require significantly more complicated analysis of the linear orders than in the first order case. This is in interesting contrast with the Laskowski-Pillay study of 'gross-models' [34]; a model is gross if every infinite definable subset of it has full cardinality. Morley's theorem can be proved in this context using the normal first order notion of type. Thus, the categoricity implies stability is routine. Intriguingly, the Laskowski-Pillay work was inspired by investigations of Moosa on the first order theory of compact complex manifolds.

The fundamental test question for the study of AEC is:

CONJECTURE 3 (Shelah's categoricity conjecture). *There is a cardinal $\mu(\kappa)$ such that for all AEC with Löwenheim number at most κ, if $\boldsymbol{K}$ is categorical in some cardinal greater than $\mu(\kappa)$ then $\boldsymbol{K}$ is categorical in all $\lambda \geq \mu(\kappa)$.*

The best approximation to the categoricity conjecture takes $\mu(\kappa)$ as the 'second Hanf number': $H_2 = H(H(\mathrm{LS}(\boldsymbol{K})))$. The initial step in the analysis [49] (see also [1]) requires the lifting to this setting of a clever integration of Morley's omitting types theorem and Morley's two cardinal theorem.

Theorem 4. [49] *Suppose $\boldsymbol{K}$ has the amalgamation property and arbitrarily large models. Suppose $\boldsymbol{K}$ is λ^+-categorical with $\lambda > H_2$. Then, $\boldsymbol{K}$ is H_2-categorical and indeed categorical on the interval* $[H_2, \lambda^+]$.

The proof requires using the omitting types theorem twice. The second time one names as many constants (H_1) as required for the first use. Categoricity on the interval is then proved by induction, making essential use of Theorem 7. Theorem 4 leads to a natural question.

Question 5. Prove or disprove. Suppose $\boldsymbol{K}$ has the amalgamation property and arbitrarily large models. Suppose $\boldsymbol{K}$ is λ^+-categorical with $\lambda > H_1$. Then, $\boldsymbol{K}$ is H_1-categorical.

In order to understand further progress on the categoricity transfer problem, we introduce an important notion (first named in [21]; the cardinal parameters were added in [4]).

Definition 6. The AEC $\boldsymbol{K}$ is (χ, λ)-(weakly) tame if for any (saturated) model M of cardinality λ, if $p, q \in \mathcal{S}(M)$ (the Galois types over M) are distinct then there is a submodel N of M with $N \leq \chi$ so that $p \restriction N \neq q \restriction N$.

Of course any first order theory is tame; i.e. $(\aleph_0, \infty)$-tame. And by [46, 47], it is consistent with ZFC that every categorical AEC defined by a sentence of $L_{\omega_1,\omega}$ is tame. But aside from the first order case (and homogeneous model theory where again every class is tame), there is no example where $(\aleph_0, \infty)$-tameness has been deduced from categoricity except as a corollary to the Morley theorem for the class. (E.g. Zilber's quasiminimal excellent classes and categorical classes in $L_{\omega_1,\omega}$ are each shown to be tame in [1]; but the result is not needed for the transfer of categoricity proof given but only an observation.)

Nontameness can arise in natural mathematical settings. An Abelian group is $\aleph_1$-free if every countable subgroup is free. An Abelian group H is *Whitehead* if every extension of Z by H is free. Shelah constructed (in ZFC) an Abelian group of cardinality $\aleph_1$ which is $\aleph_1$-free but not a Whitehead group. (See [14, Chapter VII.4].) Baldwin and Shelah [8] code this into an example of nontameness. Essentially a point codes an abelian group which is the right end of a short exact sequence; every countable approximation to the group splits but the whole group does not. Thus the AEC is not $(\aleph_0, \aleph_1)$-tame. Baldwin and Shelah [8] also show that nontameness is essentially a distinct phenomena from non-amalgamation by showing any AEC $\boldsymbol{K}$ which exemplifies a nontameness property (satisfying a mild condition) can be transformed to one which does satisfy amalgamation and still fails the tameness. But this transformation destroys categoricity and even stability. In my view, the most significant (nontrivial) sufficient condition for tameness is due to Shelah:

THEOREM 7. [49] *Suppose $\boldsymbol{K}$ has the amalgamation property and arbitrarily large models. Suppose $\boldsymbol{K}$ is λ-categorical with $\lambda > H_1$. For every κ with $H_1 \leq \kappa < \lambda$, $\boldsymbol{K}$ is (χ, κ)-weakly tame for some $\chi < H_1$.*

The combination of Shelah's downward categoricity argument and the tameness argument gives the result for 'tame' instead of 'weakly tame' if H_1 is allowed to grow to H_2. The argument for Theorem 7 in [49] is flawed. A short and correct argument due to Hyttinen, correcting and elaborating various exegises given separately by Baldwin and Shelah, appears in [1]. This result poses several questions.

QUESTION 8. Suppose $\boldsymbol{K}$ has the amalgamation property and arbitrarily large models. Suppose $\boldsymbol{K}$ is λ-categorical with $\lambda > H_1$.

1. Is there any way to reduce the upper bound on χ in Theorem 7 (or find a lower bound above $\mathrm{LS}(\boldsymbol{K})$)?
2. Is there any way to replace weakly tame by tame?
3. And most important, (compare 1.16 of [21]), can $\kappa = \lambda$ in Theorem 7?

A positive answer to Question 8.3 would yield a full solution of the categoricity problem for AEC with amalgamation and arbitrarily large models.

Is there any way to weaken the categoricity hypothesis in Theorem 7 to stability?

QUESTION 9. Suppose $\boldsymbol{K}$ has the amalgamation property and arbitrarily large models. Prove or disprove: If $\boldsymbol{K}$ is κ-stable with $\kappa > H_1$ then $\boldsymbol{K}$ is (weakly) (H_1, κ)-tame.

In the light of Theorem 4 and Theorem 7, it is interesting to examine Question 9 at the successor of the categoricity cardinal in the hypothesis of Theorem 4. How much would it help to know stability in λ^{++}?

Shelah speaks rather loosely of locality in various places. We have broken this notion into three precise concepts. Following [21], we have chosen tame as the name of one of these. We call the others locality and compactness. There is considerable to be learned about the relations among the parameterized versions of these notions; the following survey just touchs on some of the natural questions that arise. Essentially, they are a few of the many ways one might make specific the general question, 'Are there AEC which are eventually categorical without the many nice properties such as tameness, excellence, locality of the known examples?'

DEFINITION 10.

1. $\boldsymbol{K}$ has (κ, λ)-*local Galois types* if for every continuous increasing chain $M = \bigcup_{i<\kappa} M_i$ of members of $\boldsymbol{K}$ with $|M| = \lambda$ and for any $p, q \in \mathcal{S}(M)$: if $p \upharpoonright M_i = q \upharpoonright M_i$ for every i then $p = q$.
2. Galois types are (κ, λ)-*compact* in $\boldsymbol{K}$ if for every continuous increasing chain $M = \bigcup_{i<\kappa} M_i$ of members of $\boldsymbol{K}$ with $|M| = \lambda$ and every increasing chain $\{p_i : i < \kappa\}$ of members $\mathcal{S}(M_i)$ there is a $p \in \mathcal{S}(M)$ with $p \upharpoonright M_i = p_i$ for every i.

The proof of Theorem 7 is very much about tameness rather than locality.

QUESTION 11. Is there any way to replace (weakly) tame by local in Theorem 7?

The constructions in [8] that create amalgamation destroy categoricity; can this be avoided? More precisely,

QUESTION 12. Find an AEC (in a countable language) which is categorical in all uncountable powers, has the amalgamation property, and which is not $(\aleph_0, \infty)$-tame (or $(\aleph_0, \infty)$-local).

Grossberg and Kolesnikov [17] recently completed an important analysis of the relationship between excellence and tameness. They work in classes which are posited to have an independence relation analogous to forking in the first order case. They show in particular that if the class satisfies the extension property for independence, the appropriate version of stationarity and the forking has $< \lambda$-character then (λ, λ^+)-tameness and (λ, λ^+)-locality follow. Further, they show that if $\boldsymbol{K}$ is χ-excellent (under an extended definition for this context), then $\boldsymbol{K}$ is (χ, ∞)-tame. There is no simple test, such as failure of the order property in the first order case, to generate a wealth of examples of classes satisfying these hypotheses. It is not even clear that eventual categoricity yields the properties. But as with tameness, these hypotheses provide a platform on which to develop a stability theory for AEC.

A positive answer to either Question 8.1 or Question 11 would seem to require essentially new methods. The distinction between syntactic (given by a set of formulas in some logic) and semantic or Galois types (given by the ability to amalgamate embeddings or as orbits in a suitably homogeneous model) leads to a quest for further examples.

QUESTION 13. What are some AEC's which are not basically given syntactically? Which of the many examples of extended logics in [10] give rise to AEC's?

A few examples appear in [15, 7, 1], but there should be many more. Zilber's work on excellent classes raises several issues here [60, 59]. He phrases his work for certain models (those satisfying the countable closure condition) in a class defined in $L_{\omega_1,\omega}$. So the class could be described in $L_{\omega_1,\omega}(Q)$; but such a formulation is of no value for the proof. The hardest part of the argument, the verification of excellence, is in the standard vein of algebraic model theory. But here infinitary conditions are being interwoven with not only algebraic but analytic arguments. Zilber's model theoretic perspective produces an intriguing group of conjectures about the complex numbers. In particular, even a very simple case of showing the complex exponential field is 'strongly exponentially closed' in the sense of [59], has only been answered using Schanuel's conjecture and Hadamard factorization [38].

In another direction, one might try to weaken the categoricity assumption for proving tameness. The following version doesn't shed much light since we don't have any clear way at hand to verify it (aside from categoricity). Shelah called this notion rigid. I discuss this notion because a number of central steps in the analysis of categoricity in [49], existence of non-splitting extensions, (H_1, λ)-tameness, and unions of saturated models are saturated are fundamentally about AEC which are epi.

Definition 14. The AEC $\boldsymbol{K}$ is *epi* if there is an EM-template Φ such that the functor $EM(\_, \Phi)$ is an *epi*morphism from linear orders onto the models of $\boldsymbol{K}$.

For example, the core of the proof of Theorem 7 shows:

Corollary 15. *If $\boldsymbol{K}$ is epi then $\boldsymbol{K}$ is (H_1, ∞)-tame.*

Categoricity is used in the proof of Theorem 7 to get that the AEC is epi. (Of course this terminology isn't used.)

Grossberg and VanDieren [20] strengthen the hypothesis of Theorem 4 by adding (μ, ∞)-tameness for some $\mu < \lambda$ with powerful results.

Theorem 16. [21] *Suppose $\boldsymbol{K}$ has the amalgamation property and arbitrarily large models. Suppose $\boldsymbol{K}$ is λ and λ^+ categorical for some $\lambda > \mathrm{LS}(\boldsymbol{K})$ and is (μ, ∞)-tame for some $\mu < \lambda$. Then $\boldsymbol{K}$ is categorical in all cardinals above λ.*

There are a number of variations on this result and on the elimination of the (categorical in λ)-hypothesis [36, 20, 7, 23, 55] to get 'upwards categoricity from a single cardinal'. We don't go into this further here except to remark that any full proof from these hypotheses involves an intensive investigation of EM models to show that a union of a short chain of saturated models is saturated [49, 1]. Natural extensions, which remain open as far as I know, are to replace categoricity in a single successor cardinal by categoricity in a regular or an arbitrary cardinal; a different idea is needed to replace the role of two cardinal models.

The stability spectrum theorem is fundamental for the study of first order theories; it is the essence of the classification of theories. But no similar result is known for general abstract elementary classes. The stability spectrum of an AEC $\boldsymbol{K}$ is the function from cardinals to cardinals which gives the supremum of the cardinals of the number of Galois types over a model in $\boldsymbol{K}$ of fixed cardinality.

Question 17. Is the stability spectra of an abstract elementary classes (even in a countable language with $\mathrm{LS}(\boldsymbol{K}) = \omega$) one of a finite set of functions? Does ω-stable imply stable in all cardinals?

Baldwin, Kueker, and VanDieren [6] give a positive response to the last question but only under the extremely strong hypotheses of both $(\aleph_0, \infty)$-tameness and $(\aleph_0, \infty)$-locality. Grossberg and VanDieren earlier noted in [21] (they state stronger hypotheses):